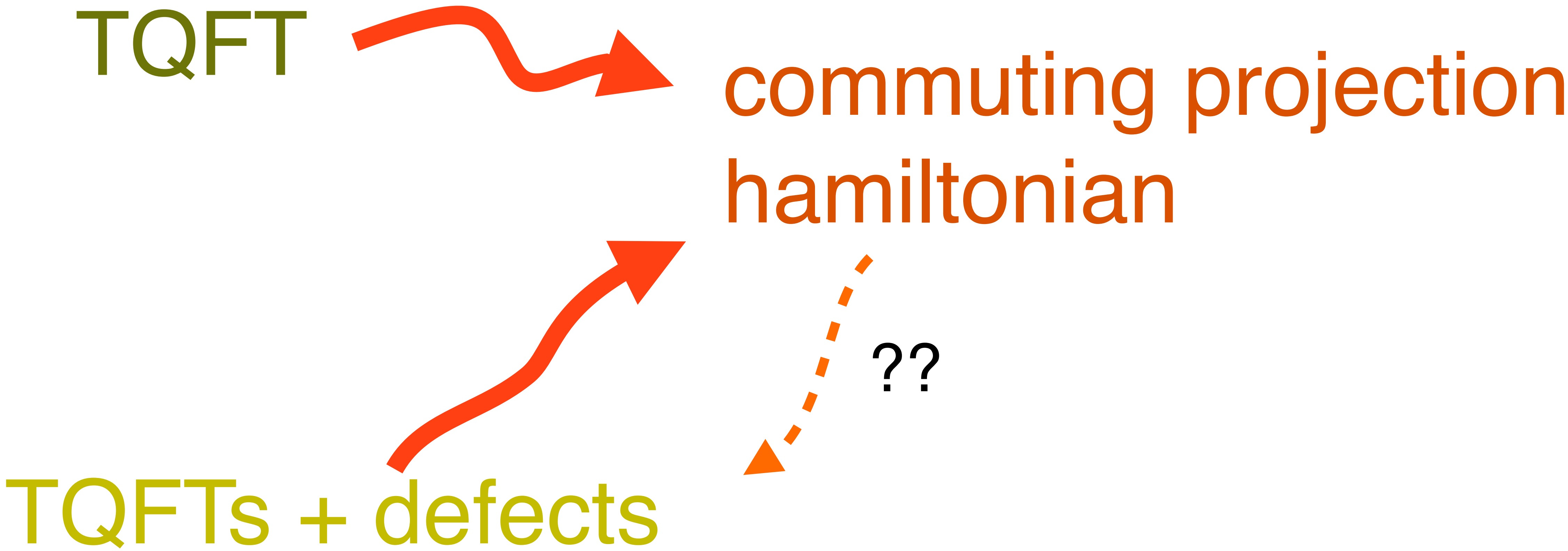
Codimension 1 defects, categorified group actions, and condensing fermions

(some parts of this talk are joint work with Z. Wang)

These slides available at http://canyon23.net/math/talks/


TQFT
commuting projection hamiltonian
??
TQFTs + defects

Outline

1. General theory of constructing a CPH from a TQFT

2. Illustrate general theory in familiar examples
    - Dijkgraaf Witten TQFT --> Kitaev finite group model (any dimension)
    - 2+1 dimensional TQFT (Turaev-Viro) --> 2d Levin-Wen model
    - 3+1 dimensional premodular TQFT --> 3d premodular (WW) model
    - Rep(G) theory --> high dimensional string net model (any dimension)

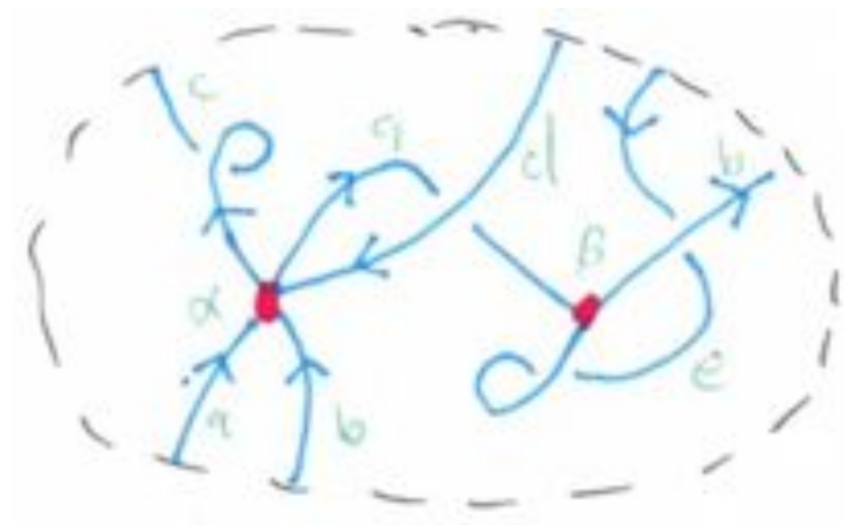
3. G-Rep(G) duality --> duality between premodular WW model and modular WW model with G action (SET/SPT)

4. Categorified group actions on trivial theories --> more SPT; relation to twisted Dijkgraaf-Witten theory

5. Condensing fermions via codimension 1 defect with Spin structure --> super Levin-Wen model

General theory (omitting most of the details)

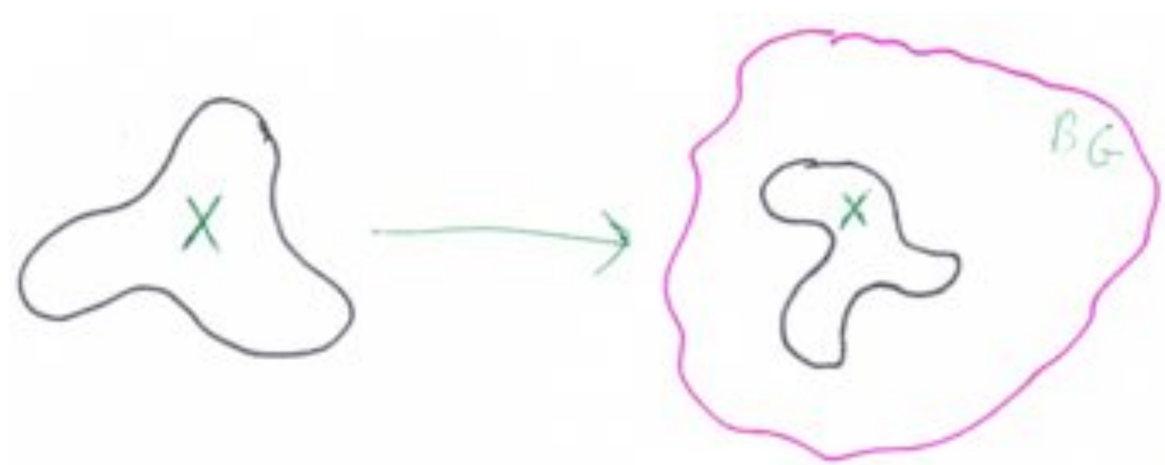
n-category: very roughly, a system for drawing pictures in n dimensions

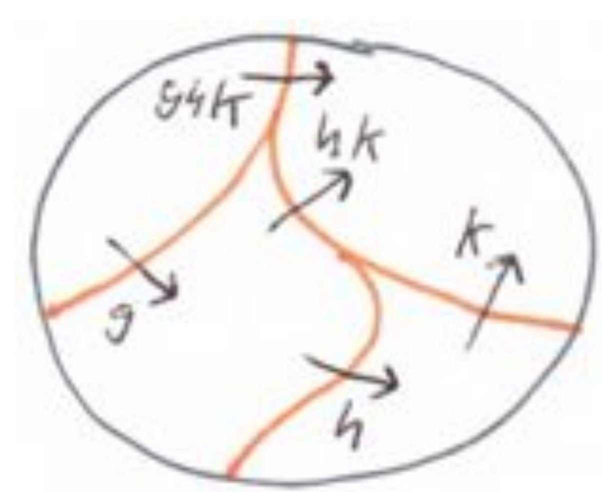


F(X) - “fields” or “pictures” on *X*

Main examples:

- $\mathcal{F}(X) = \{$all maps from X to $BG\}$ (or $\{$flat G bundles over $X\}$ or $\{G$-foam pictures drawn on $X\}$)
- $\mathcal{F}(X) = \{$generalized string nets drawn on $X\}$



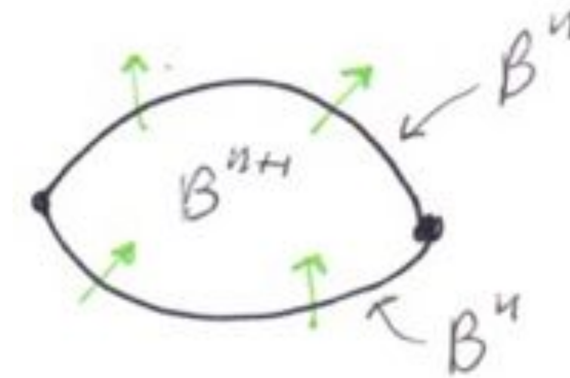


An n-category is the local part of an n+1-dimensional TQFT; an n+1-dimensional TQFT is a global version of an n-category

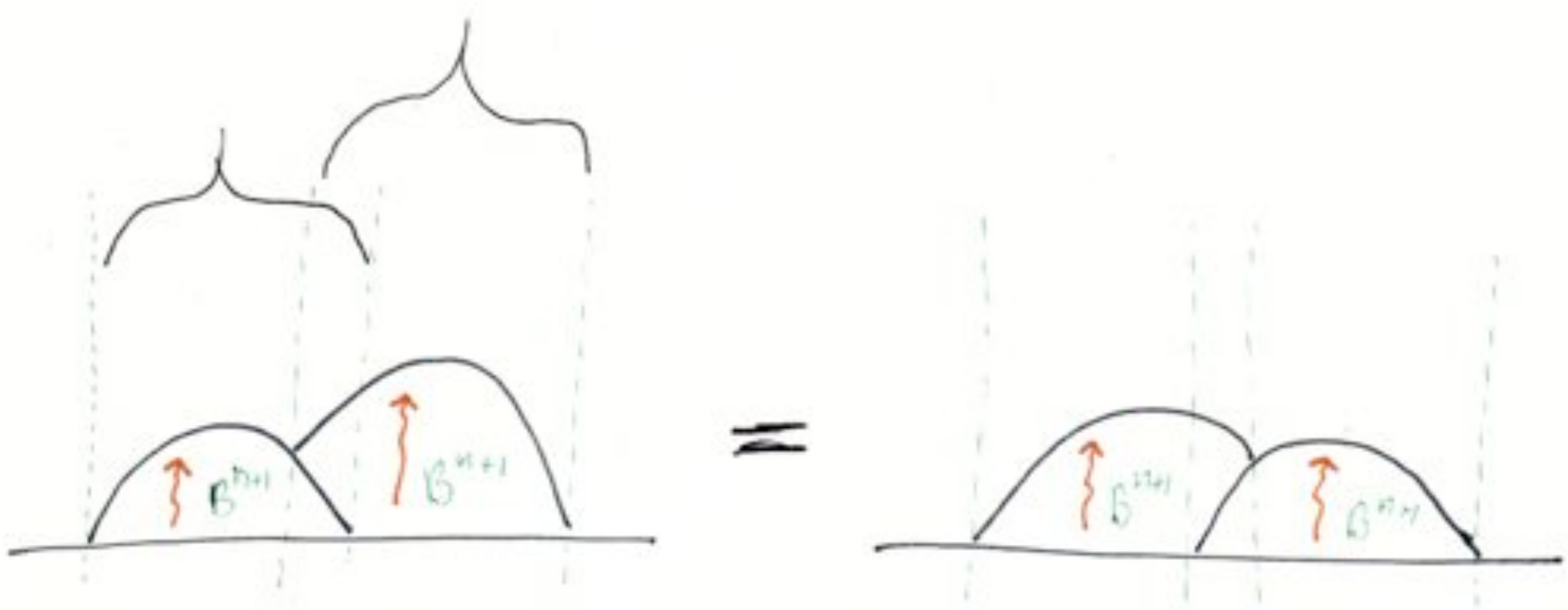
A TQFT has two basic building blocks: the **fields** and **local projections** (dually, **local relations**). Everything else is built from these ingredients.

- The local projections are one version of the path integral of the n+1-ball

$$Z(B^{n+1}) : Z(B^n) \to Z(B^n)$$



Local projections commute, so TQFTs have (continuous, non-lattice) commuting projections built in from the start.

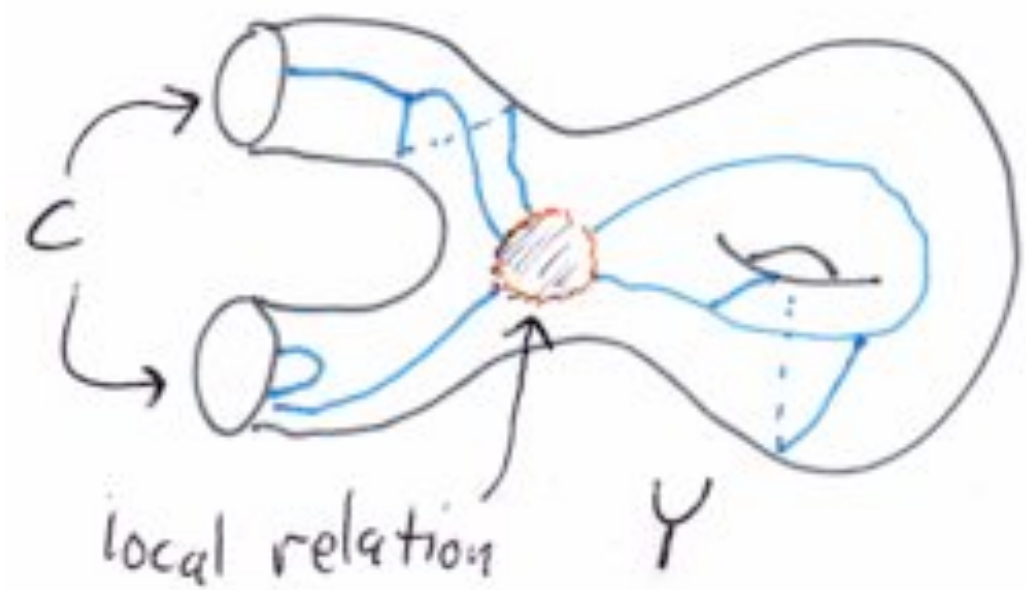


Two basic constructions

Hilbert space

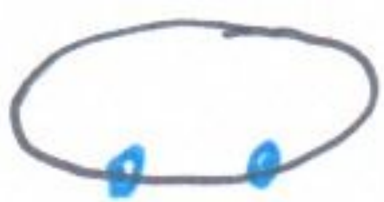
- For Y an n-manifold and c a boundary condition in $\mathcal{F}(\partial Y)$, define the (predual) Hilbert space $A(Y;c)$ fo be finite linear combinations of fields in $\mathcal{F}(Y;c)$ modulo local relations.

We will see in a moment that this Hilbert space can be identified with the ground state of an associated commuting projection hamiltonian



Cylinder category

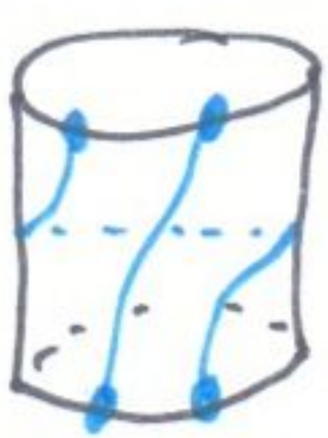
- For Y an $(n-1)$-manifold and $c \in \mathcal{F}(\partial Y)$, define $A(Y;c)$ to be the 1-category



objects : $\mathcal{F}(Y;c)$

morphisms $a \to b$: $A(Y \times I; a, b)$

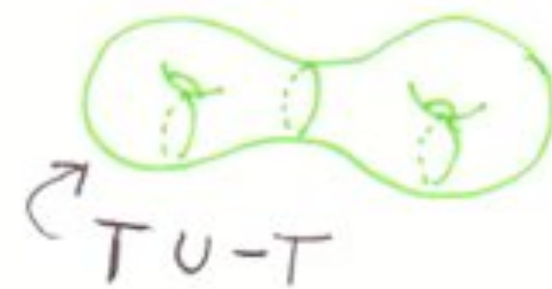
composition : stacking

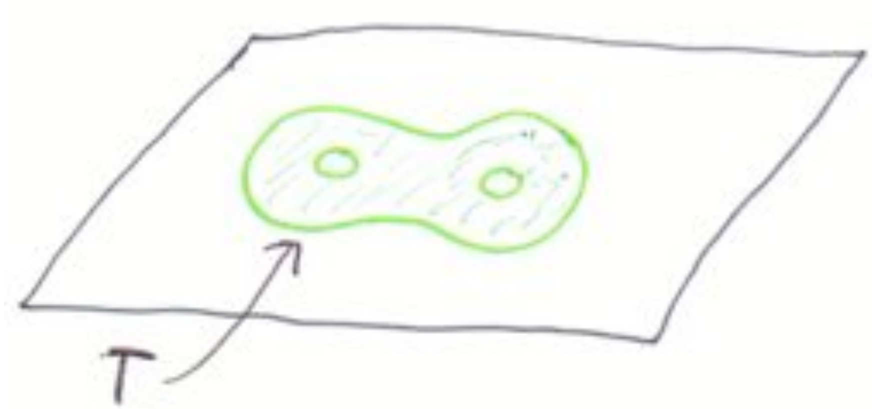


Without analyzing the hamiltonian, we know:

- $\{\text{Quasiparticles with boundary } S\} \longleftrightarrow \{\text{irreps of } A(S)\}$

- $\{\text{Operators with footprint } T\} \longleftrightarrow A(T \cup -T)$

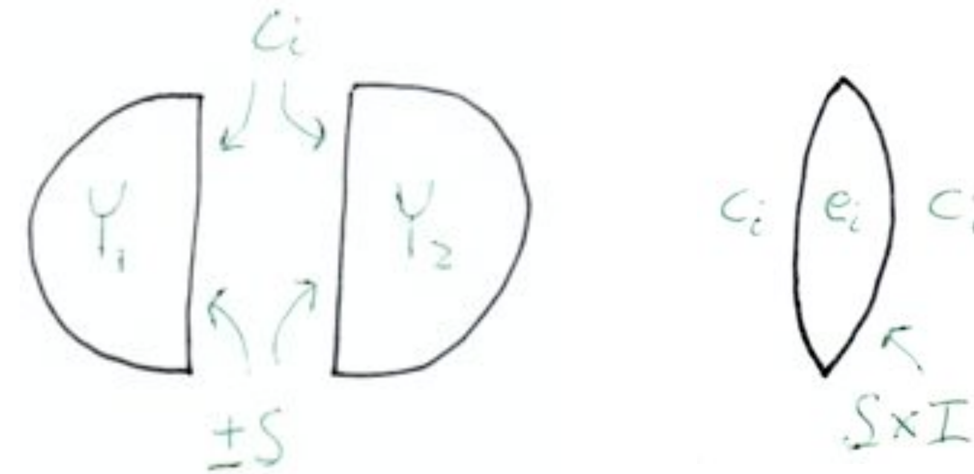




Local description of the Hilbert space

Commuting projection hamiltonian

First consider cutting into two pieces



- Standard results tell us

$$A(Y_1 \cup_S Y_2) \cong A(Y_1) \otimes_{A(S)} A(Y_2) \quad \cong \left[\bigoplus_{c \in \mathcal{F}(S)} A(Y_1; c) \otimes A(Y_2; c) \right] \Big/ \langle ax \otimes b \sim a \otimes xb \rangle$$

- If $A(S)$ is semisimple, then the above simplifies to

$$A(Y_1 \cup_S Y_2) \cong \bigoplus_i \pi_{e_i} \left(A(Y_1; c_i) \right) \otimes \pi_{e_i} \left(A(Y_2; c_i) \right),$$

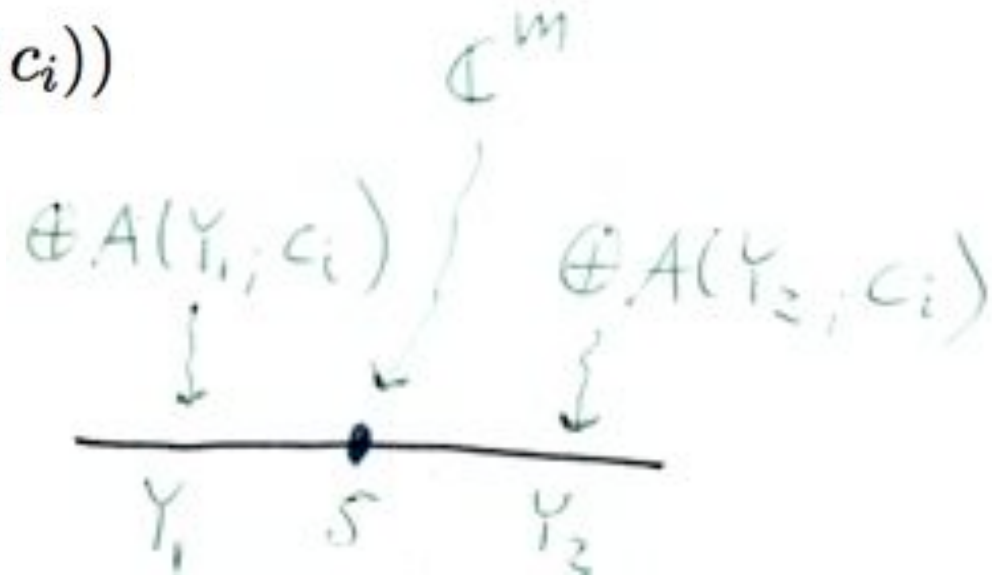
  where

$$\{e_i : c_i \to c_i\}, \;\; 1 \leq i \leq m$$

- Note that now everything is defined in terms of Hilbert spaces defined entirely on one side or the other, plus some projections.

- We can reinterpret this as the ground state for a commuting projection hamiltonian...

$$A(Y_1 \cup_S Y_2) \cong \bigoplus_i \pi_{e_i}\left(A(Y_1; c_i)\right) \otimes \pi_{e_i}\left(A(Y_2; c_i)\right)$$

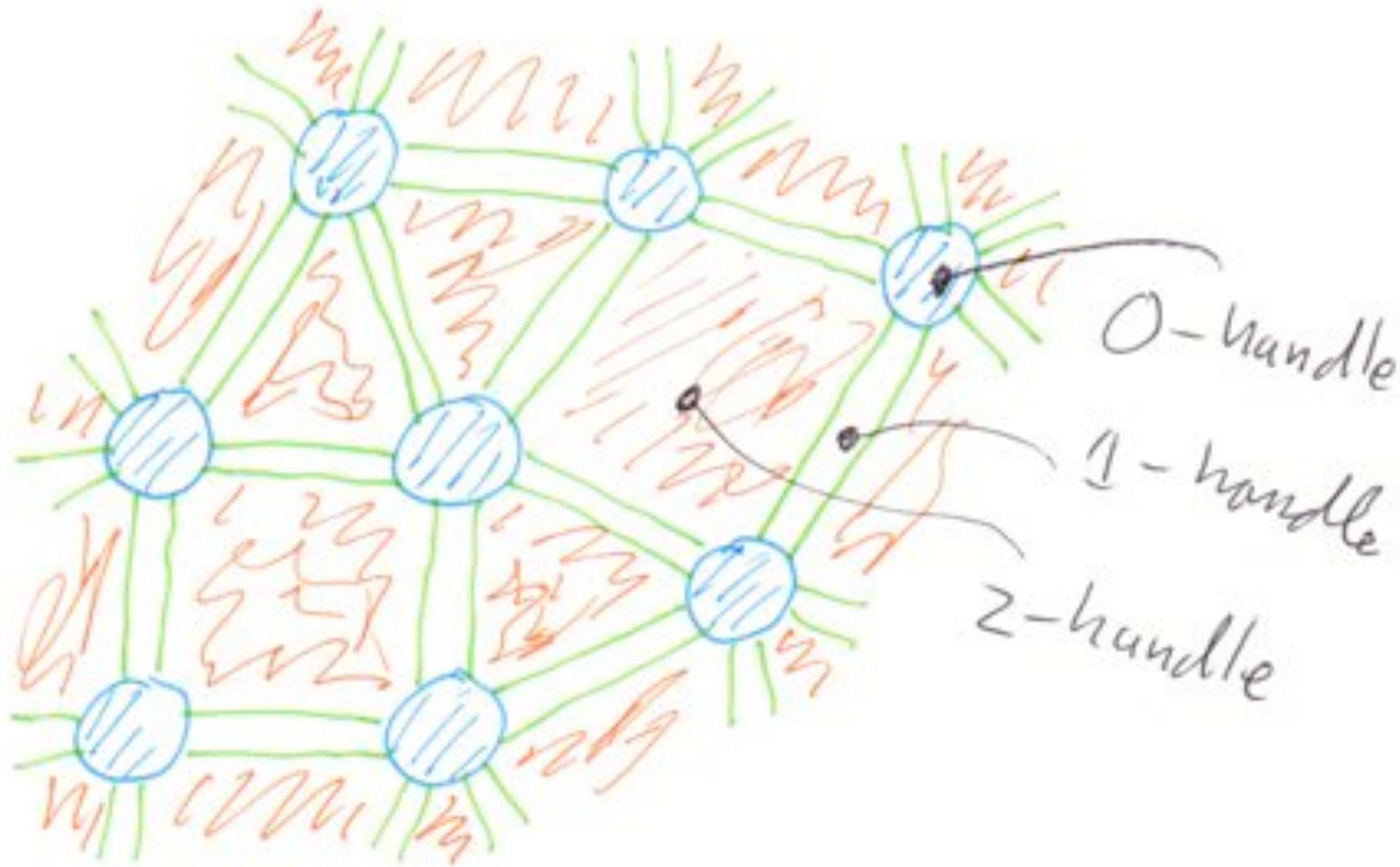


- We have spins (degrees of freedom) $\mathbb{C}^m$ associated to the cut S, and $\bigoplus_i A(Y_j; c_i)$ associated to Y_j.

- For lack of better nomenclature, I'll call $\mathbb{C}^m$ "idempotent" degrees of freedom and $A(Y_j; c_i)$ "vertex" degrees of freedom. Note that $\mathbb{C}^m$ has a preferred basis corresponding to the idempotents, while the choice of basis of $A(Y_j; c_i)$ might be arbitrary.

- The hamiltonian has two sorts of projections: "idempotent" projections corresponding to π_{e_i} above, and "bookkeeping" projections which ensure that the indices i match up and $A(Y_j; c_i)$ is non-zero.

- If $A(Y_j; c_i)$ is 1-dimensional, then can ignore that vertex degree of freedom. Similarly, if $A(Y_j; c_i) = 0$ for all but one i, then we can ignore the corresponding idempotent degree of freedom.



Applying the above procedure repeatedly to handle decomposition of Y produces a lattice commuting projection hamiltonian.

All of the specific hamiltonian examples in this talk come from applying the above procedure. In other words, this is a uniform way of deriving all of these hamiltonians.

Familiar examples...

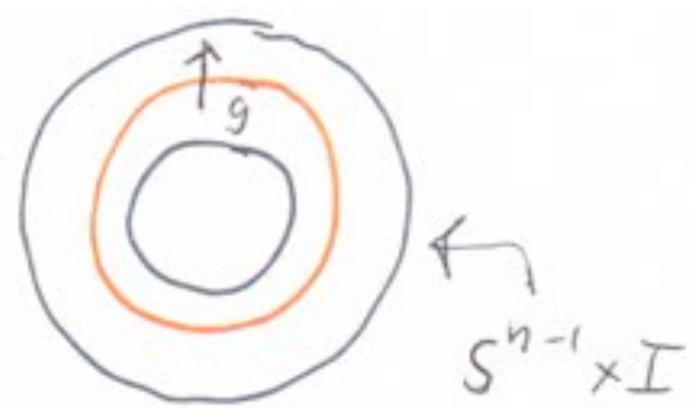
1. Dijkgraaf-Witten theory for finite group G

Kitaev finite group model

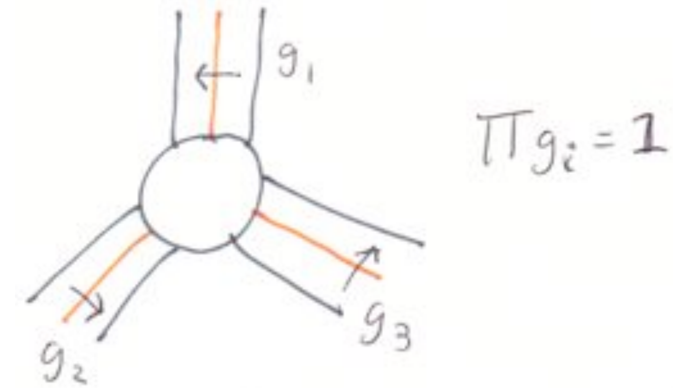
- For attaching n-handles, the relevant category is $A(S^{n-1})$, which is Morita equivalent to the group algebra $\mathbb{C}[G]$ when $n \geq 3$.
- The only idempotent in $A(S^{n-1})$ which is non-zero in the n-handle is

$$\frac{1}{|G|}\sum_{g\in G} s_g,$$

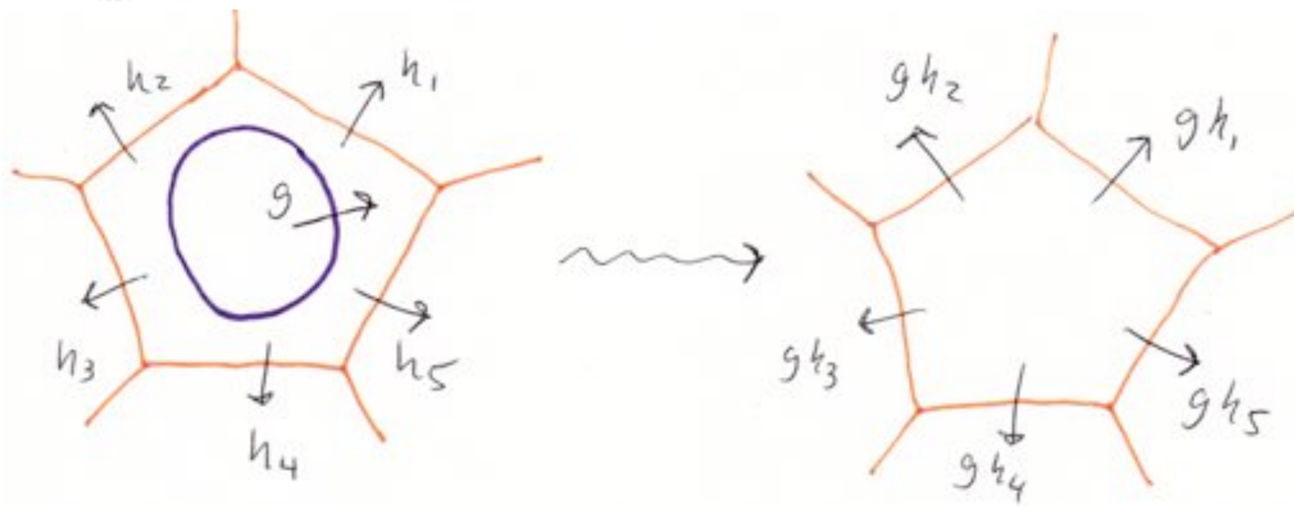
so attaching n-handles leads to terms in the hamiltonian corresponding to this idempotent and no degrees of freedom.

$S_g =$ 

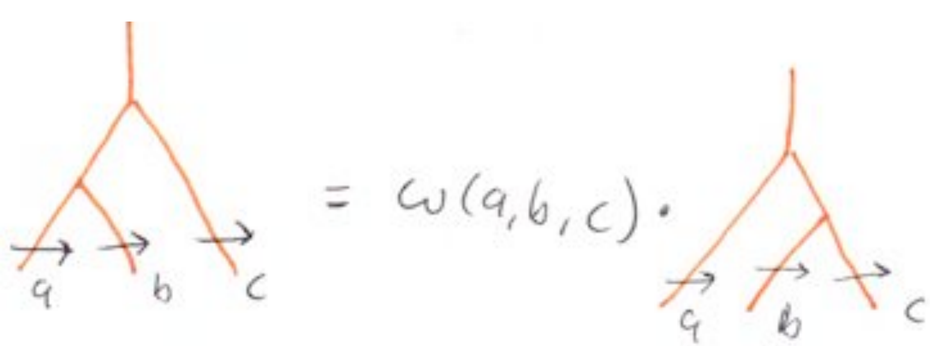
- For attaching $(n-1)$-handles, the relevant category is $A(S^{n-2}\times I;\emptyset)$. This category has a (very simple) idempotent for each $g \in G$. So we get degrees of freedom $\mathbb{C}^G$ for each $(n-1)$-handle.
- The $(n-2)$-handles lead to bookkeeping restrictions $\prod_i g_i = 1$ around each $(n-2)$-handle.
- None of the higher codimension handles contributes anything interesting to the hamiltonian.

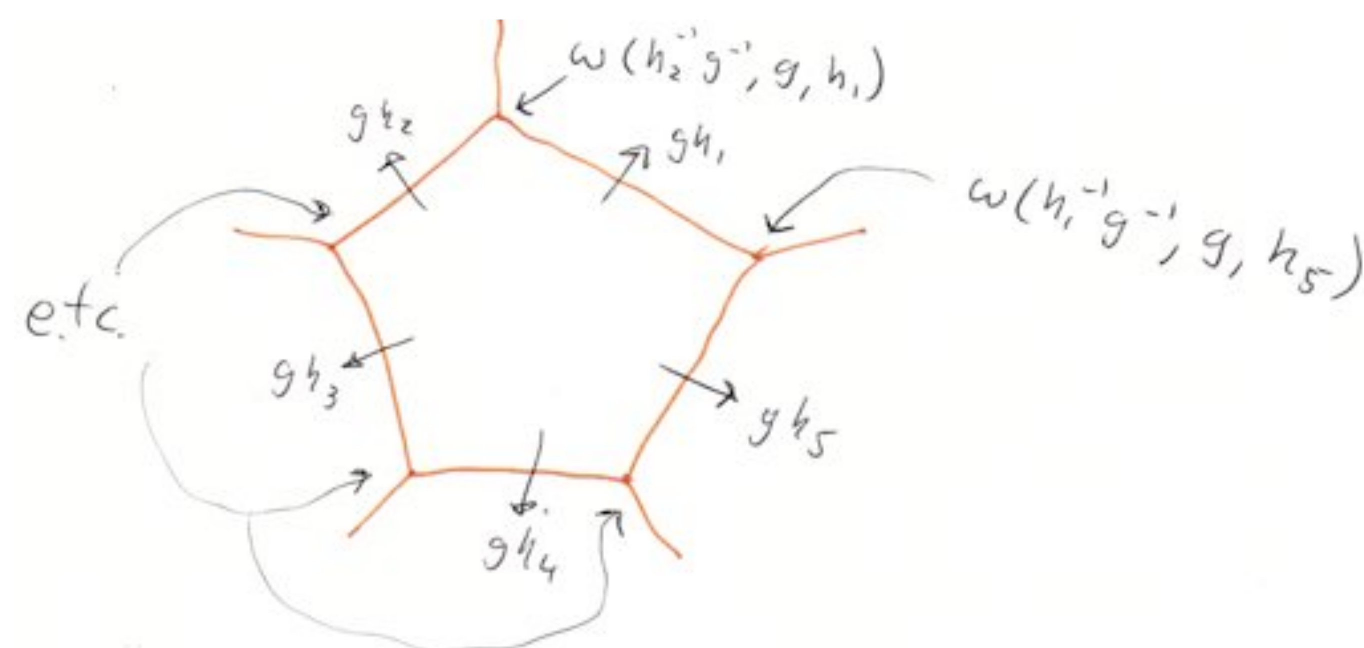


- The operator s_g acts as follows:



- If we have a symmetric, normalized cocycle $\omega \in C^{n+1}(BG, U(1))$, we can twist the hamiltonian.

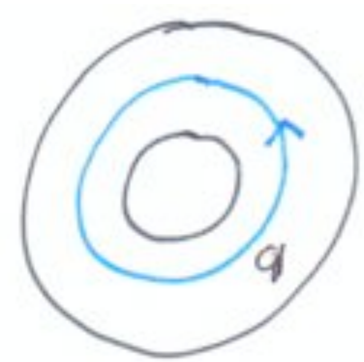




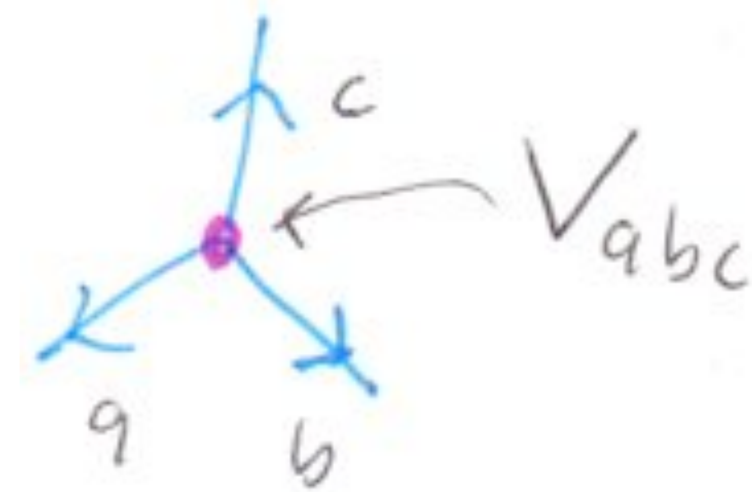
2. Turaev-Viro TQFT (generic 2+1-dimensional TQFT)

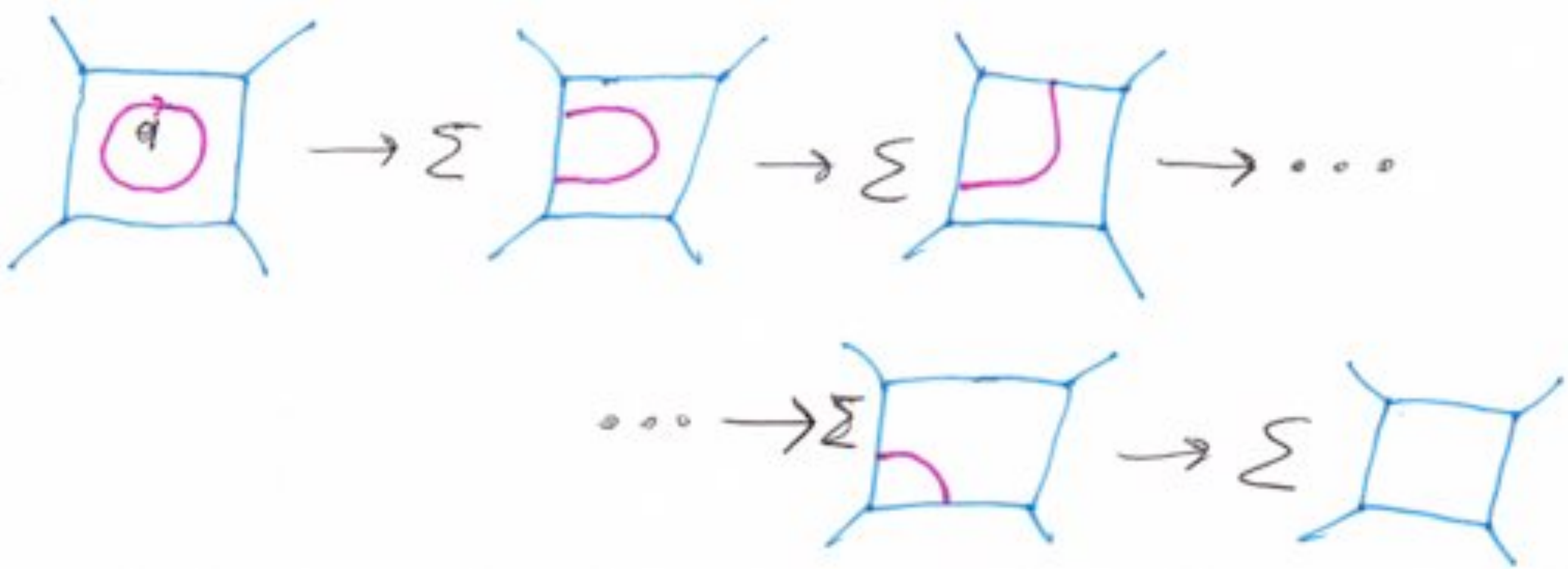
2d Levin-Wen hamiltonian

- The input for a Turaev-Viro TQFT is a strict pivotal 2-category (e.g. tensor category) C.
- For attaching 2-handles, the relevant category is $A(S^1)$. The only idempotent in $A(S^1)$ which is non-zero in the 2-handle is
$$\frac{1}{D}\sum_a d_a l_a.$$
(The sum is over simple objects of C, d_a is the quantum dimension, and $D = \sum_a d_a^2$.)

$\ell_a =$ 

- So attaching 2-handles leads to terms in the hamiltonian corresponding to this idempotent and no degrees of freedom.
- For attaching 1-handles, the relevant category is $A(I;\emptyset)$. This Morita equivalent to C as a 1-category and has a (very simple) idempotent for each simple object a of C. So we get degrees of freedom $\mathbb{C}^m$ for each 1-handle, where m is the number of simple objects.
- The 0-handles contribute additional degrees of freedom V_{abc}. We get bookkeeping terms in the hamiltonian when $V_{abc} \cong 0$.

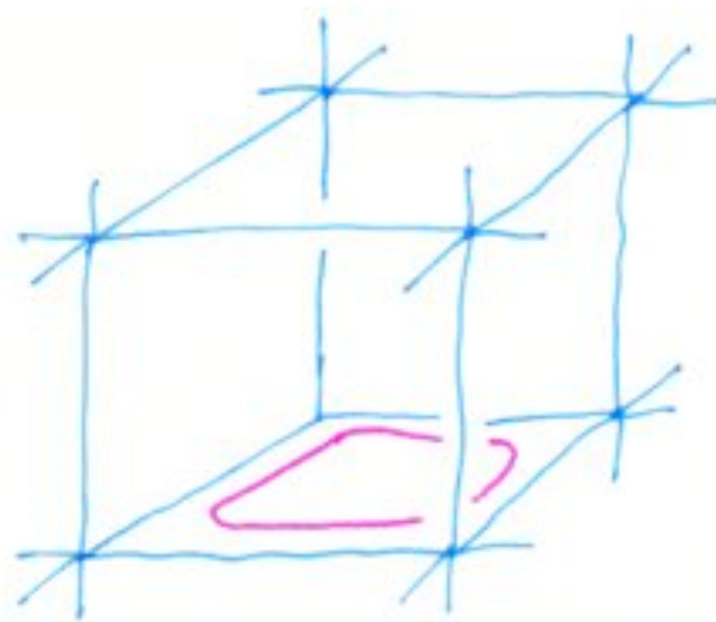




3. Premodular 3+1-dimensional TQFT

3d premodular (WW) hamiltonian

- The input for a premodular TQFT is a premodular category (spherical braided unitary fusion category) C.
- The 3-handles contribute nothing interesting to the hamiltonian.
- The 0-, 1- and 2-handles play roles very similar those of the Levin-Wen model. The only complication is that we must take braidings into account when computing the action of the loop operator l_a.



- Let $T \subset C$ is the transparent subcategory generated by objects which braid trivially is everything. Either $T \cong \mathrm{Rep}(G)$ or $T \cong \mathrm{Rep}(G, \alpha)$, where G is a finite group and (G, α) is a super group.

- C is modular if and only if T is trivial. In this case $A(M)$ is 1-dimensional for any closed 3-manifold M. So modular categories are almost trivial.

- But not quite trivial: they give rise to non-trivial 4-manifold invariants (signature) and non-trivial central extensions of the mapping class groups of surfaces. Additionally, one can extract all of the information of a Chern-Simons (2+1)-dimensional TQFT from the modular (3+1)-dimensional TQFT.

- Theorem: If C is modular then $A_C(S^1)$ is Morita equivalent to the trivial 2-category. (This accounts for the almost-triviality of C.)

- Conjecture: In the general case where C is premodular, $A_C(S^1)$ is Morita equivalent to $A_T(S^1)$ as 2-categories. (In the case where $T \cong \mathrm{Rep}(G)$, this would imply that the C theory is very similar to (3+1)-dimensional Dijkgraaf-Witten theory for G.)

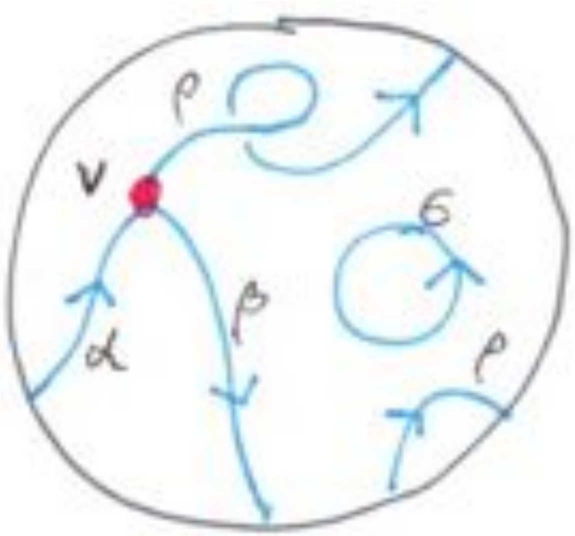
4. Rep(G) n+1-dimensional TQFT (any n)

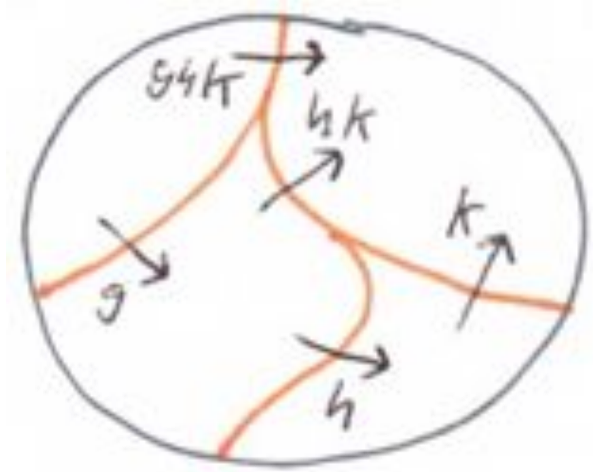
n-dim'l Rep(G) hamiltonian (name?)

- The input is the symmetric monoidal category $R = \mathrm{Rep}(G)$.
- The handles of index 3 and higher contribute nothing interesting to the hamiltonian.
- The 0-, 1- and 2-handles play roles very similar those of the Levin-Wen and premodular models. As in the premodular model, we must take braidings into account when computing the action of the loop operator l_a.

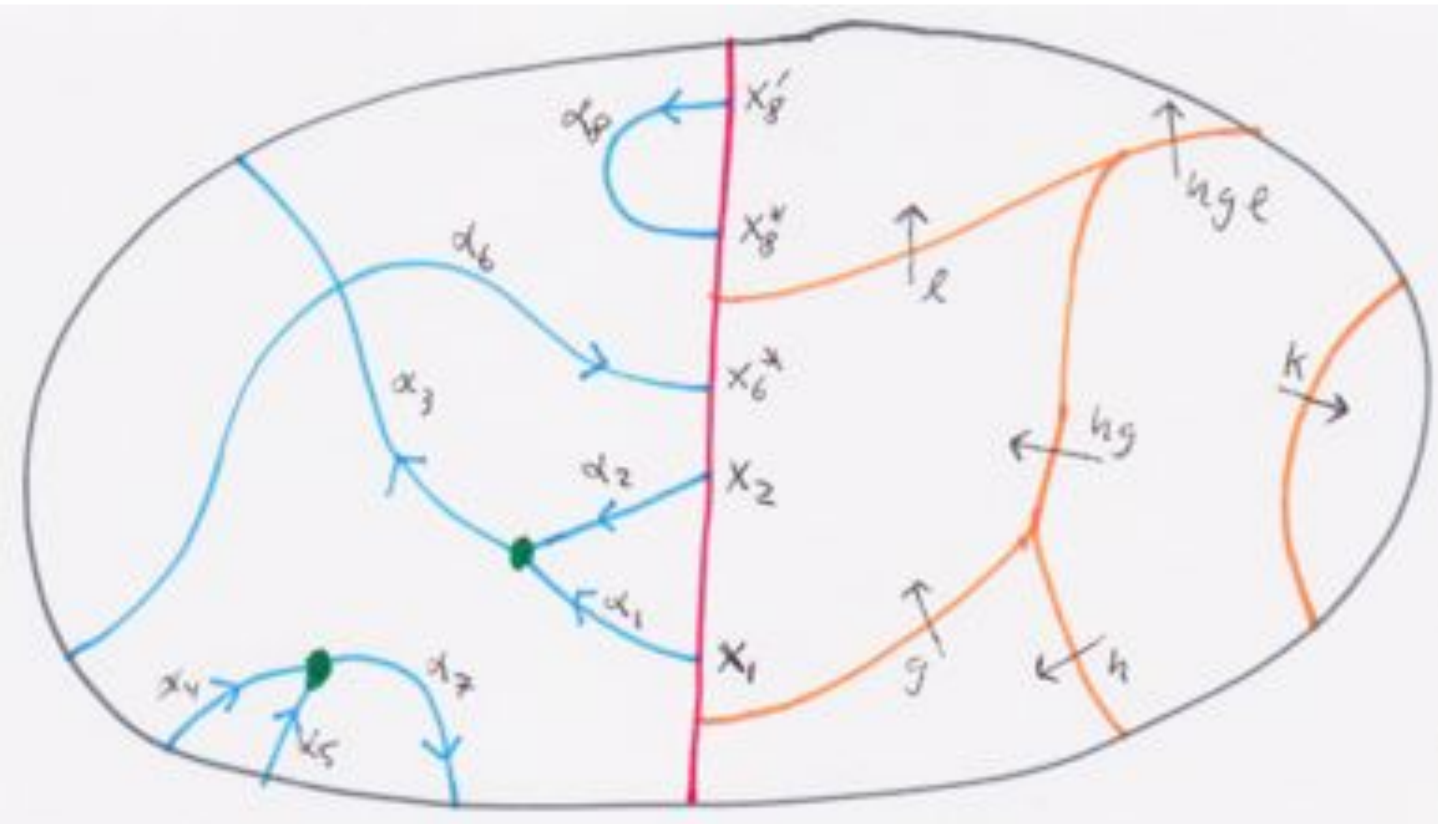
G-Rep(G) duality and 3d SET/SPT

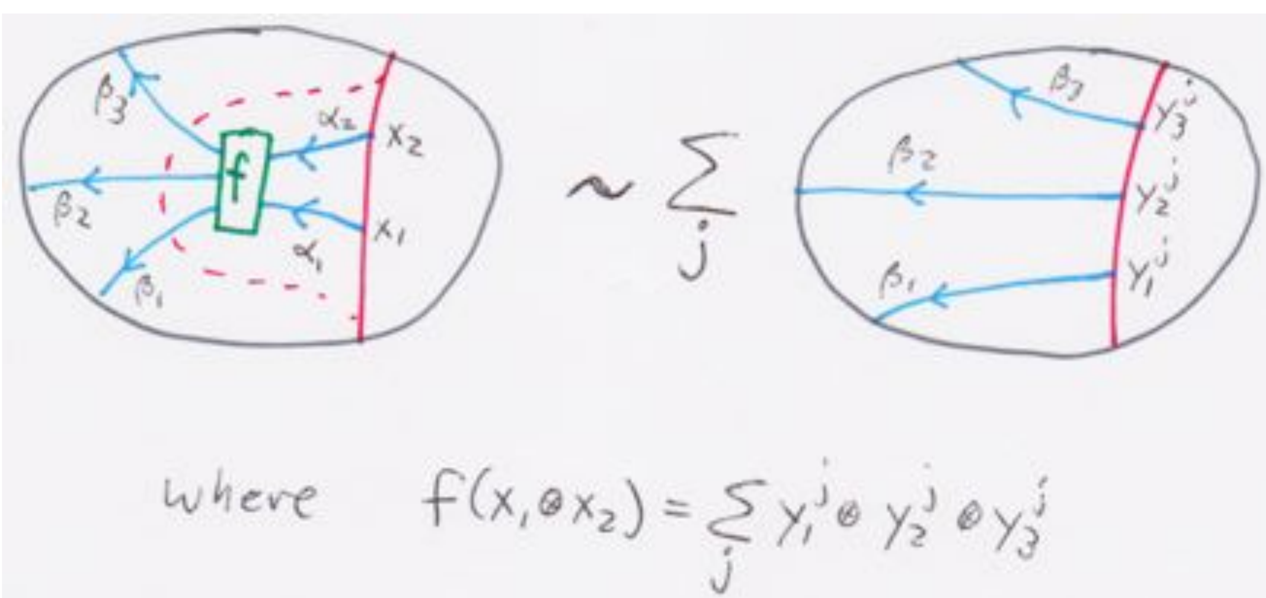
- We can construct n-categories from both G and $R = \mathrm{Rep}(G)$. For G, the non-trivial morphisms are all in degree 1. For R, the non-trivial morphisms are in degrees $n-1$ and n.
- These two n-categories are not functor-equivalent, but we will see that they are Morita equivalent.



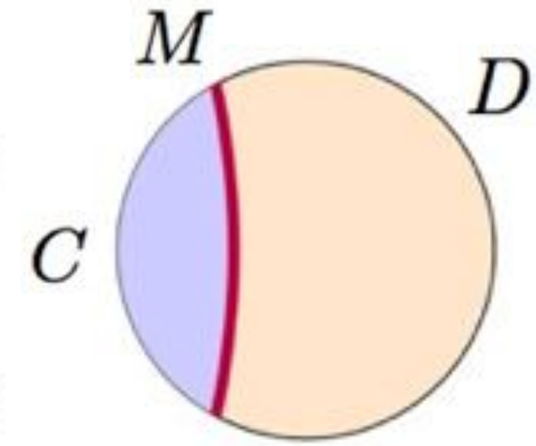


- Define a bimodule (codimension-1 defect) F between G and R as follows. (F stands for Fourier transform.)





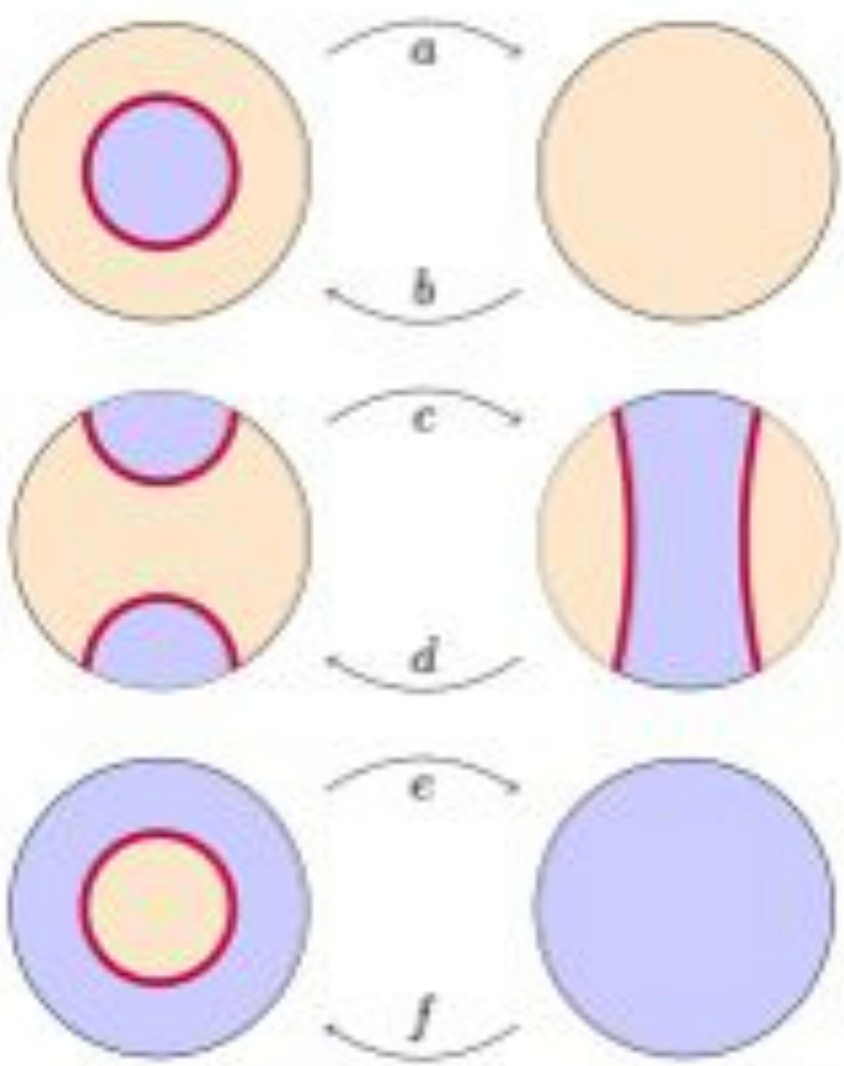
Morita equivalence for n-categories with duality

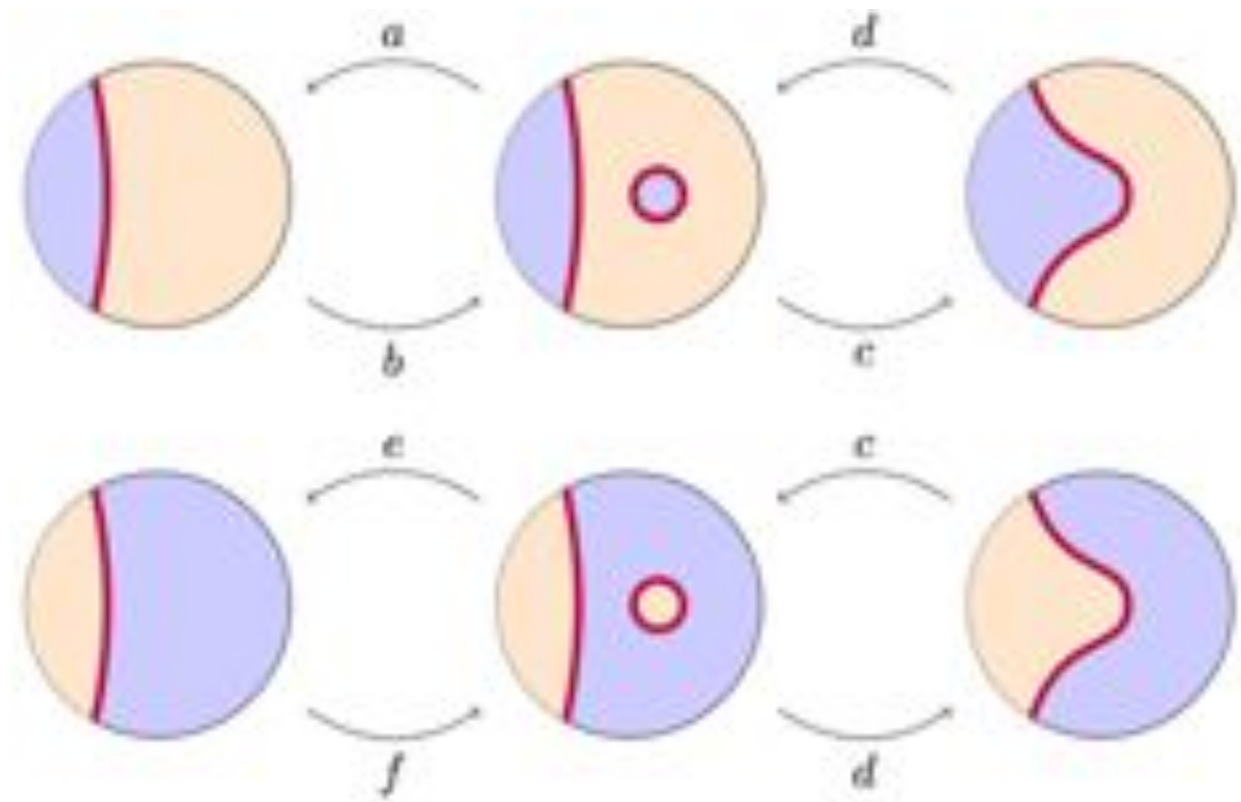


Definition. A Morita equivalence between n-categories C and D consists of

- a C-D bimodule M
- isomorphisms for each local "Morse move" on C-D-colored n-manifolds

This data is required to satisfy a relation for each possible Morse cancellation.





- Morita equivalent n-categories give rise to equivalent $(n+\epsilon)$-dimensional TQFTs. More specifically, for any closed $(n-k)$-manifold Y, $A_C(Y)$ and $A_D(Y)$ are Morita equivalent k-categories.

- Note that because of duality we specify much less data than in a typical Morita equivalence definition.

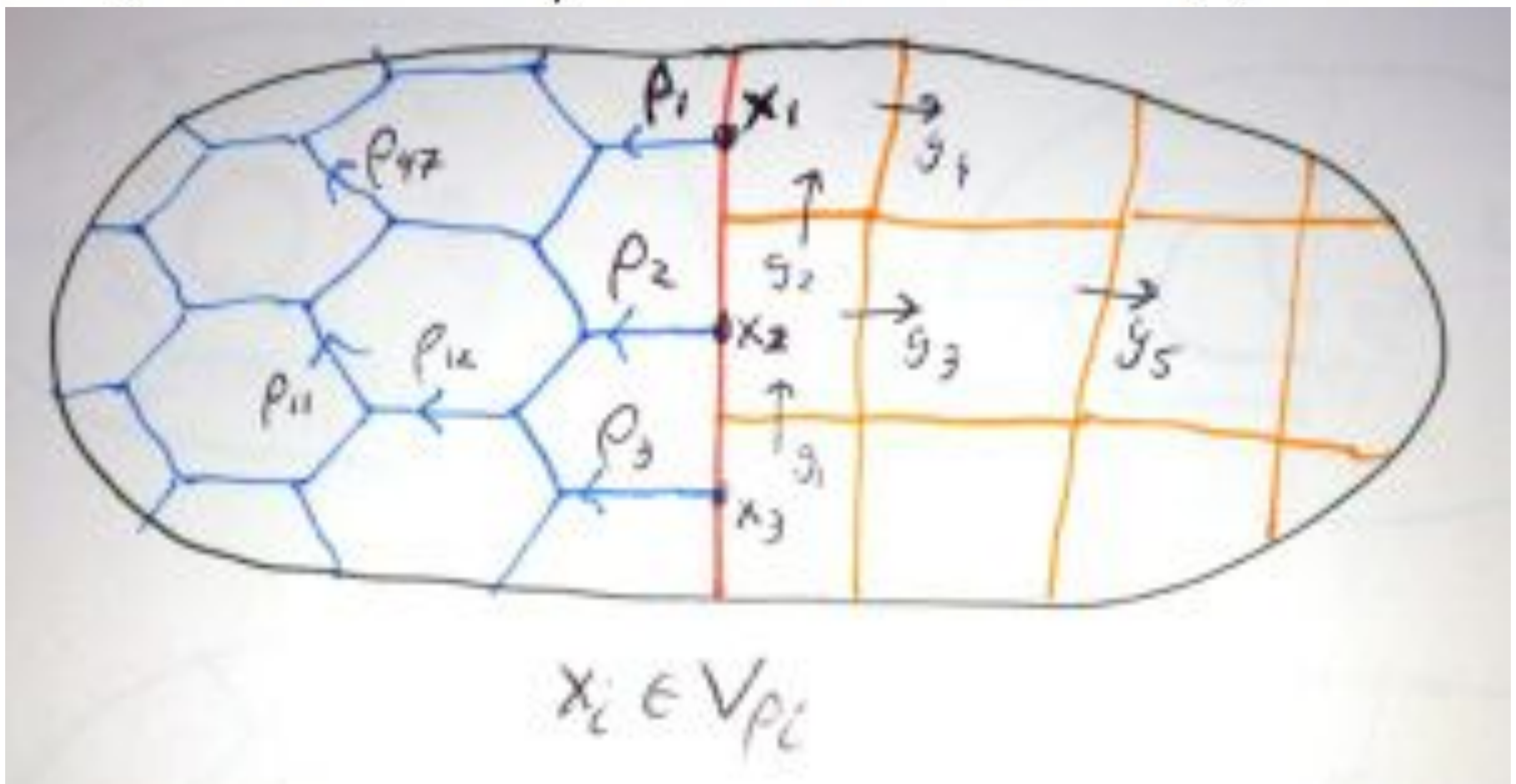
- If we equip the bimodule M with an inner product, we can extend the definition of Morita equivalence so that it yields equivalent $(n+1)$-dimensional TQFTs.

Theorem. The bimodule F_{m-1} is a Morita equivalence between G_m and R_m (for any m).

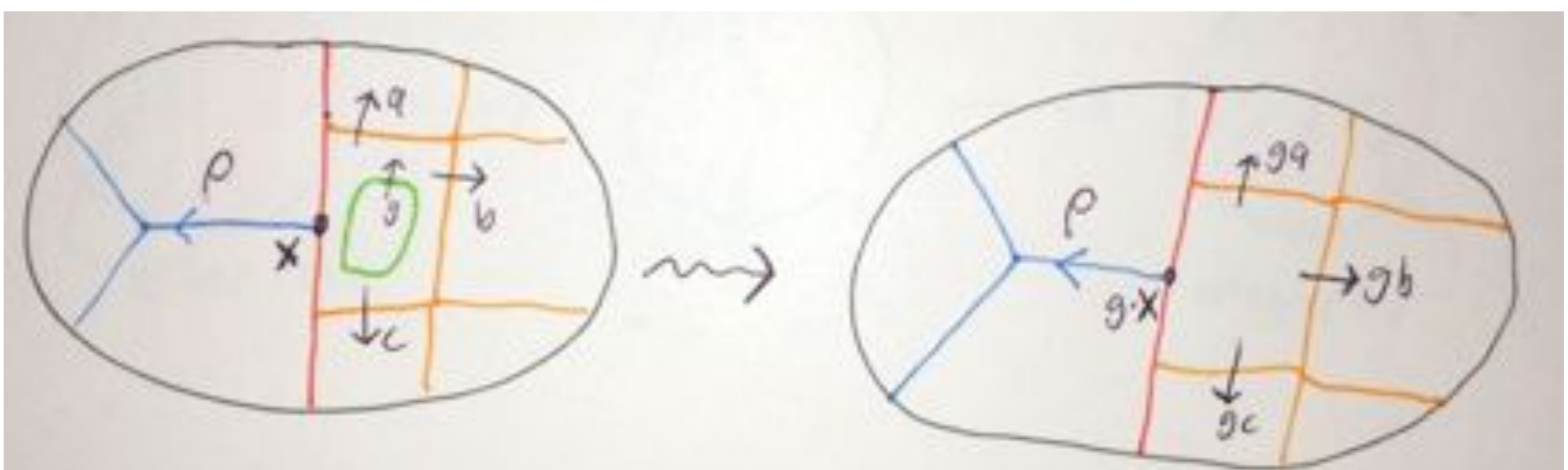
The proof boils down to the familiar fact that $\mathbb{C}[G] \cong \bigoplus \mathrm{End}_{\mathbb{C}}(\rho)$, both as commutaive algebras and $\mathbb{C}[G]$-$\mathbb{C}[G]$ bimodules.

- Using the standard procedure, we can define a commuting projection hamiltonian for n-manifolds subdivided into G parts and R parts, with the codimension-1 defect F separating the two.

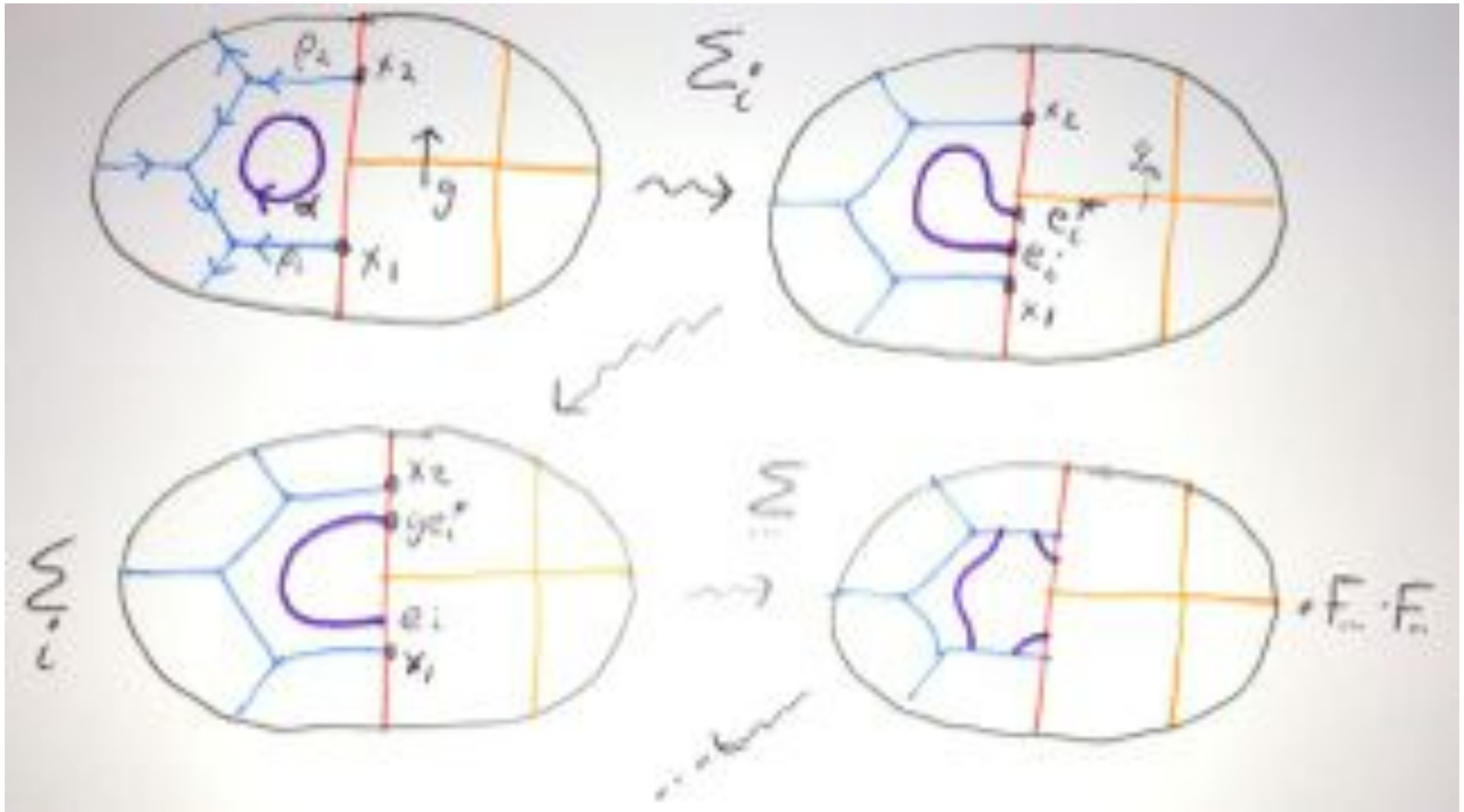
- There are new degrees of freedom V_ρ where an R-arc labeled by ρ meets the F wall.



- The n-cell terms of the G hamiltonian acts as follows near F. (Same idempotent, different action.)

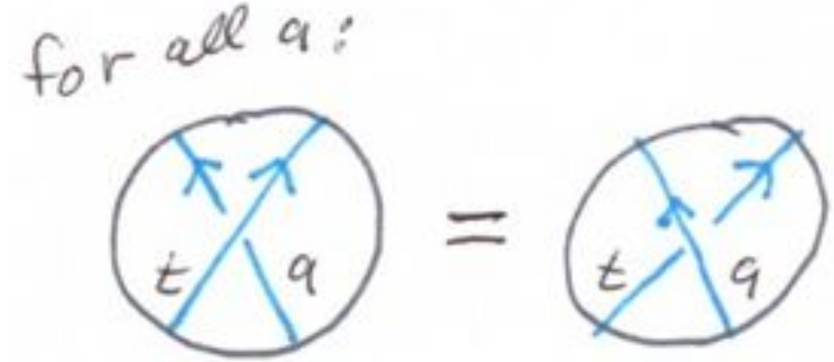


- The 2-cell terms of the R hamiltonian acts as follows near F. (Same idempotent, different action.)

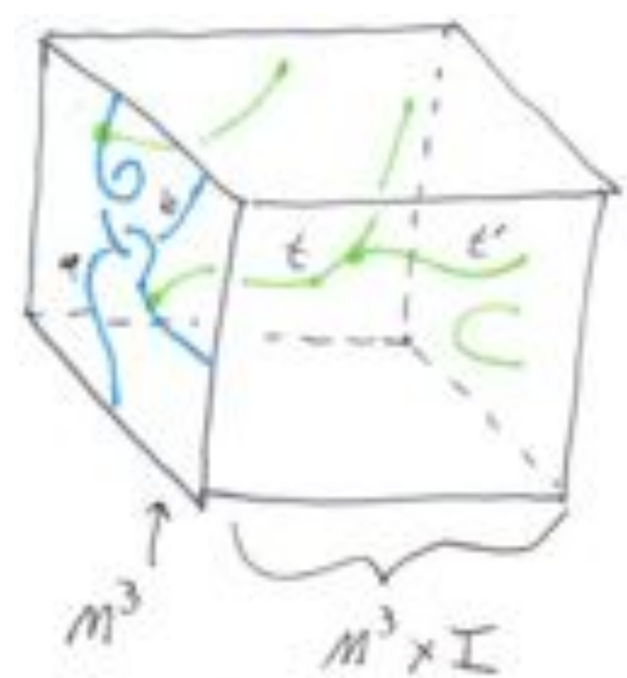


- When $n = 2$ and G is abelian, $G \cong R$ (but not canonically). In this case pushing a quasi-particle through the F domain wall exchanges electric and magnetic particles.

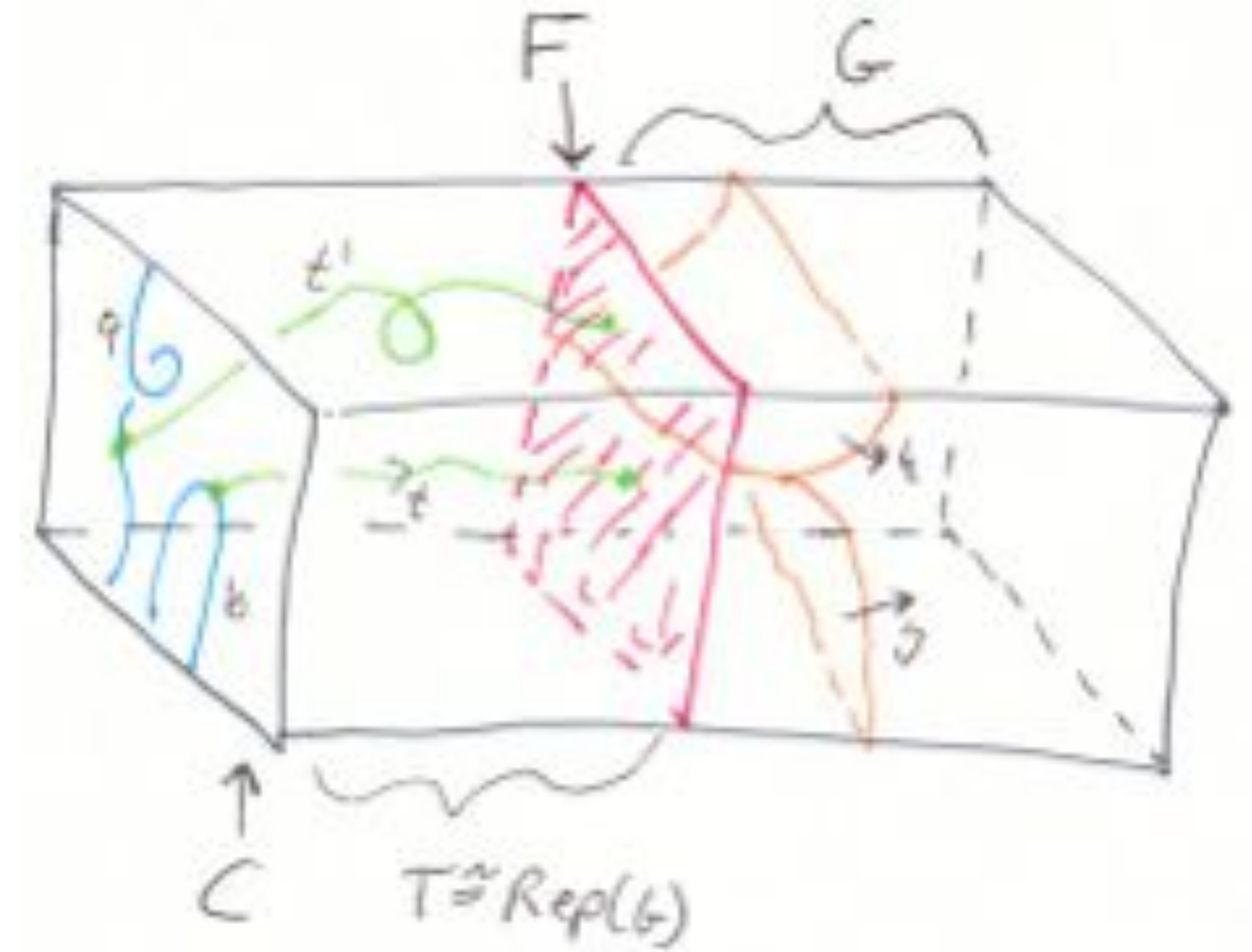
- Recall our the modular category C and the transparent subcategory T. Because T is symmetric monoidal, we can think of it as an m-category for any m, and in particular for $m = 4$. Let T_4 denote the 4-category version of T.



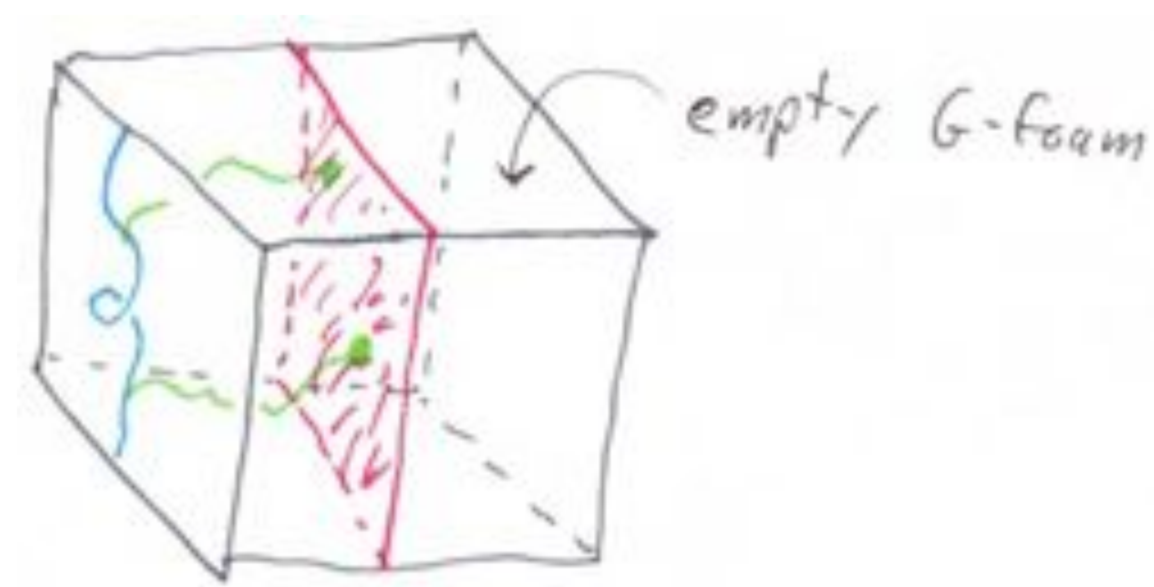
We can think of the 3-category C as a module for the 4-category T_4. Equivalently, we can think of C as a domain wall between T_4 and the vacuum.

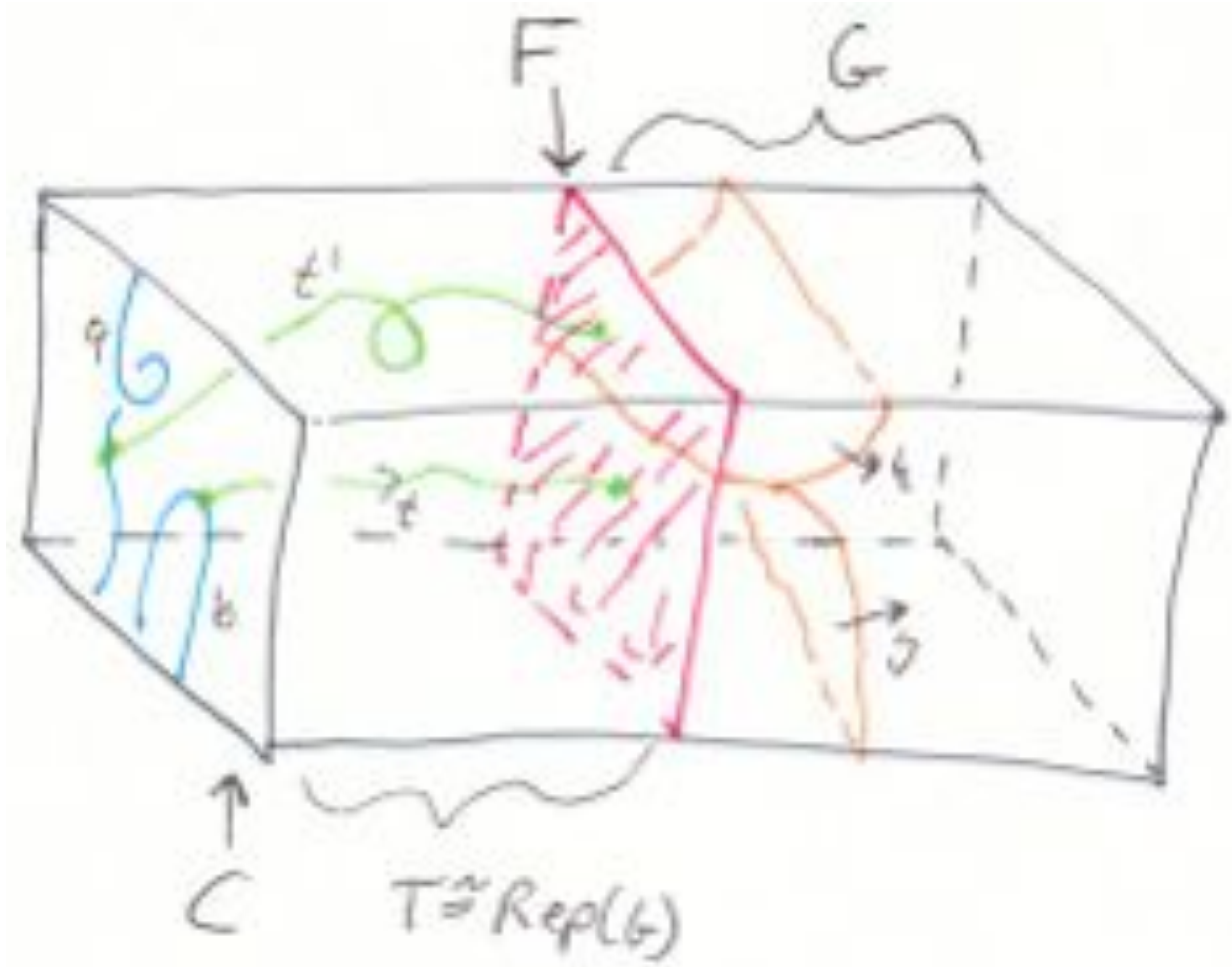


- Recall that $T \cong \mathrm{Rep}(G)$ or $T \cong \mathrm{Rep}(G, \alpha)$. For simplicity, assume that $T \cong \mathrm{Rep}(G)$. We can now apply the domain wall (bimodule) F.
- This gives us a new 3-category $C \otimes_T F$ which has a G_4 action.



It is not hard to show that $C \otimes_T F$ (ignoring the G_4 action) is modular. Tensoring with F killed off all the transparent objects in T.





- Because F is a Morita equivalence (invertible domain wall), this sets up a duality

$$\begin{array}{c}\text{premodular categories with}\\ \text{transparent subcategory} \cong \mathrm{Rep}(G)\end{array} \longleftrightarrow \begin{array}{c}\text{modular categories with}\\ \text{a } G_4 \text{ action}\end{array}$$

cf. Burnell, Chen, Fidkowski, Vishwanath

SET/
SPT

- This is very closely related to the equivariantization/deequivariantization construction of Müger and Bruguières. For example, there is a natural isomorphism

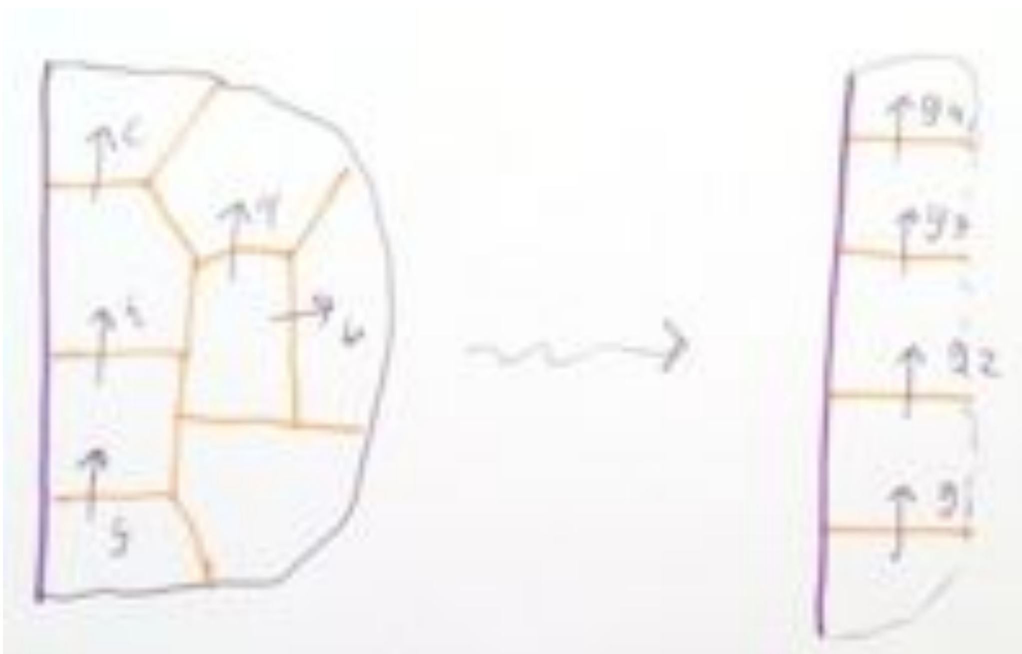
$$\mathrm{Rep}(C \otimes_R F) \cong \mathrm{Equivariantization}(\mathrm{Rep}(C)).$$

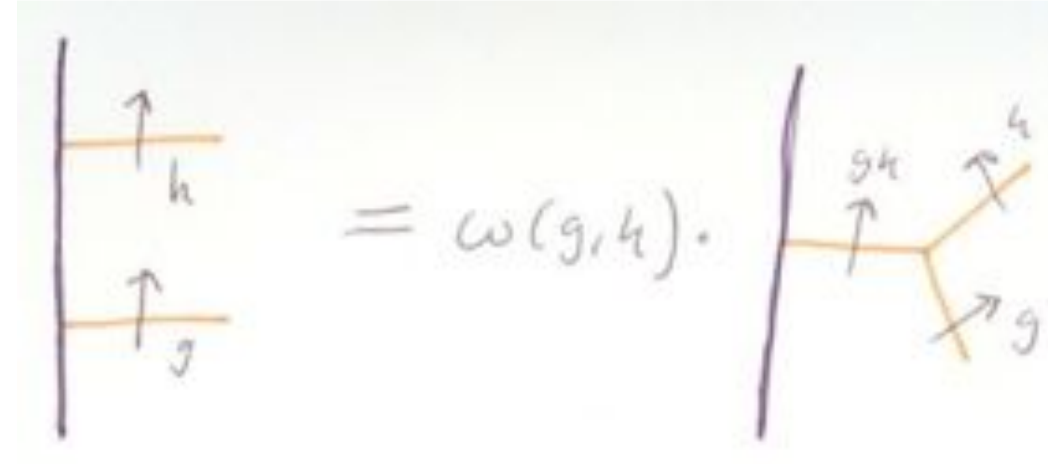
Categorified group actions on trivial theories

- Note that having an action of G_m on an $(m-1)$-category is a very general way of saying that the finite group G acts on C. It is equivalent to having a flat connection on a bundle of $(m-1)$-categories over the classifying space BG.
- Such a connection assigns an $(m-1)$-category to each point of BG, a functor to each 1-cell of BG, a 1st order natural transformation to each 2-cell of BG, a 2nd order natural transformation to each 3-cell of BG, and so on up to m-cells. The flatness condition concerns the $(m+1)$-cells.

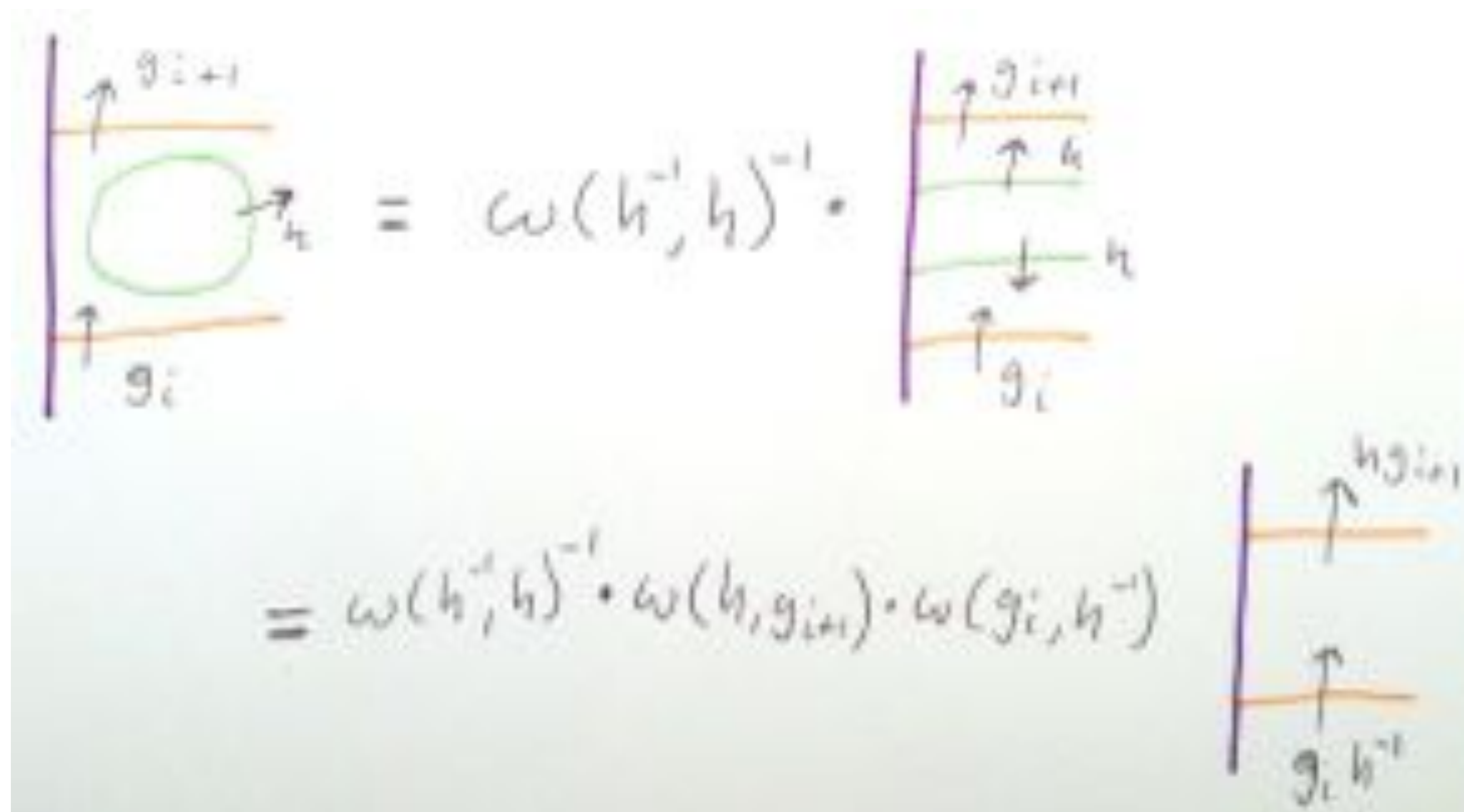
- Special case: A G_m action on the trivial linear $(m-1)$-category is exactly the same thing as a cocycle in $C^m(BG, U(1))$.

- The associated hamiltonian:





cf. Gu, Wen, ...



- Recall that given a cocycle in $C^m(BG, U(1))$ we can construct a twisted $(m-1)$-dimensional Dijkgraaf-Witten theory.

- Key observation: twisted $(m-1)$-dimensional Dijkgraaf-Witten theory is a module for untwisted m-dimensional Dijkgraaf-Witten theory; we can think of twisted $(m-1)$-dimensional Dijkgraaf-Witten theory as a codimension-1 defect between untwisted m-dimensional Dijkgraaf-Witten theory and the vacuum.

- Put another way: If we **gauge out** the G_m symmetry on the trivial $(m-1)$-category (described by some m-cocycle), the result is twisted $(m-1)$-dimensional Dijkgraaf-Witten theory (described by that same cocycle).

n-category

n-dimensional topological phase

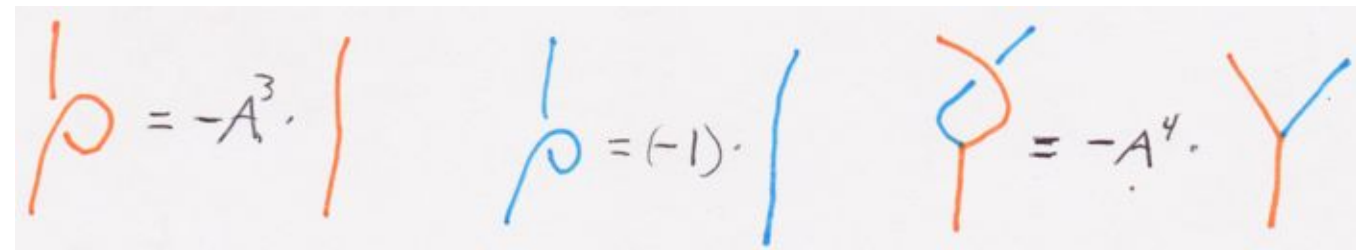
n-category with (categorified) G action

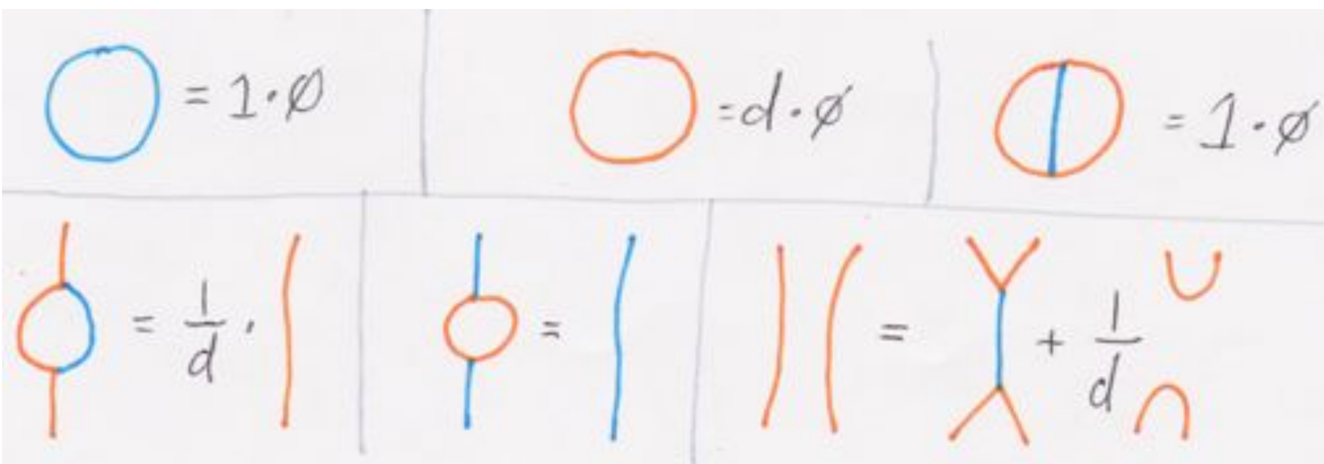
boundary theory for untwisted n+1-dimensional DW theory

n-dimensional SET

Defects with spin structures and condensing fermions

- Recall the Ising TQFT (evil twin of $SU(2)_2$).



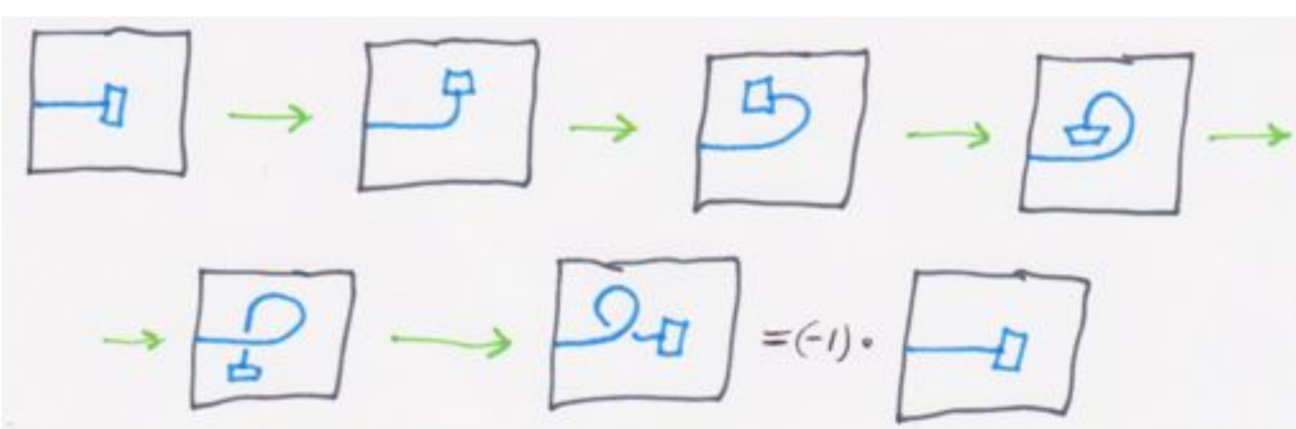


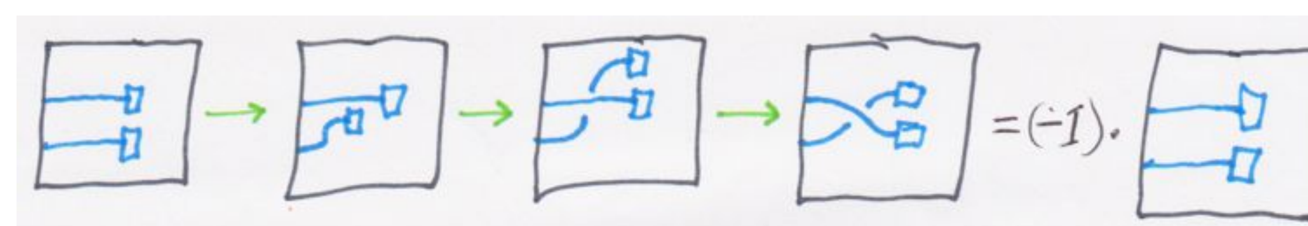
- The 1 and ψ particles form a subcategory. We would like to condense ψ, i.e. add morphisms to our category which force $\psi \cong 1$.
- But ψ is fermionic (in two different senses), so this will not be straightforward.

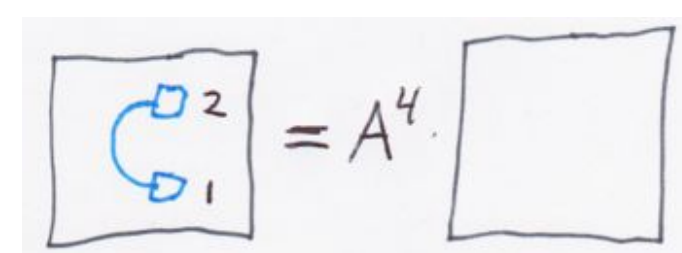
- Also, ψ is not transparent, so we expect that after condensing we will have a 2D system rather than a 3D system. (Actually, we'll get a 2.5D system in some sense.)

- We will allow ψ particles to appear from or disappear into a designated "back wall" (codimension-1 defect to the vacuum).

- Because of the fermionic nature of ψ, the back wall, W, will have to carry some state information which cancels out the factors of -1 associated to ψ. More specifically, we will need a flat line bundle over the configuration space of framed ψ-endpoints in W.

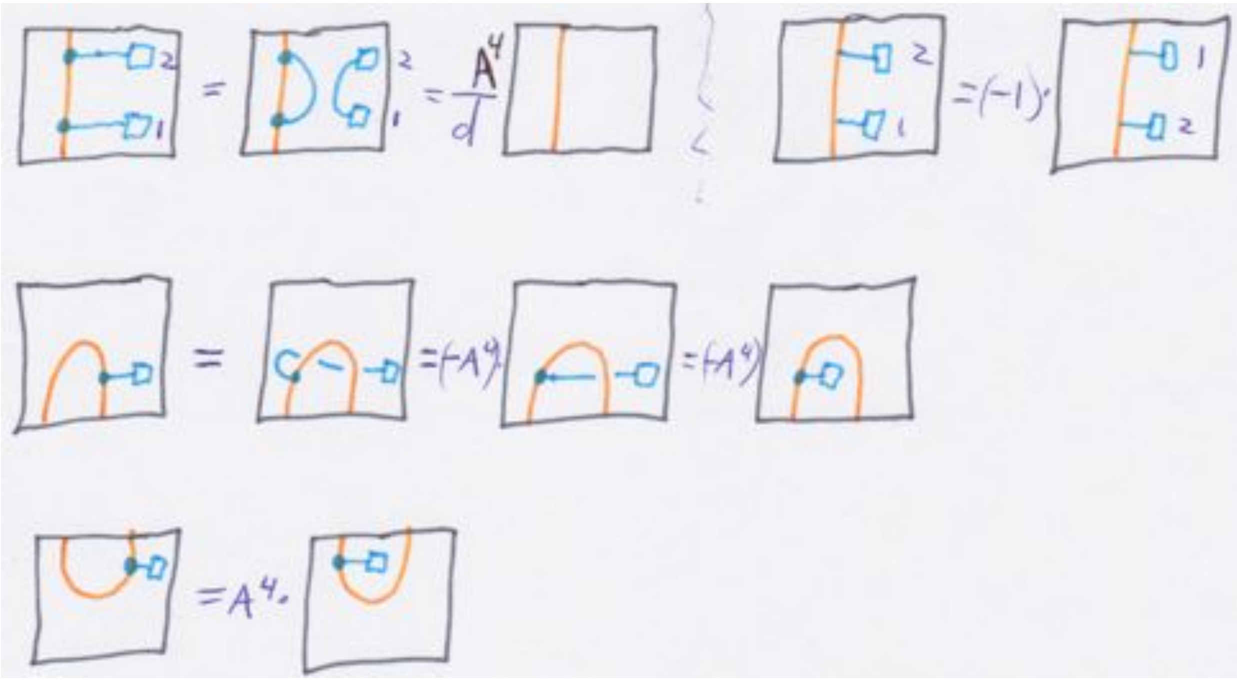


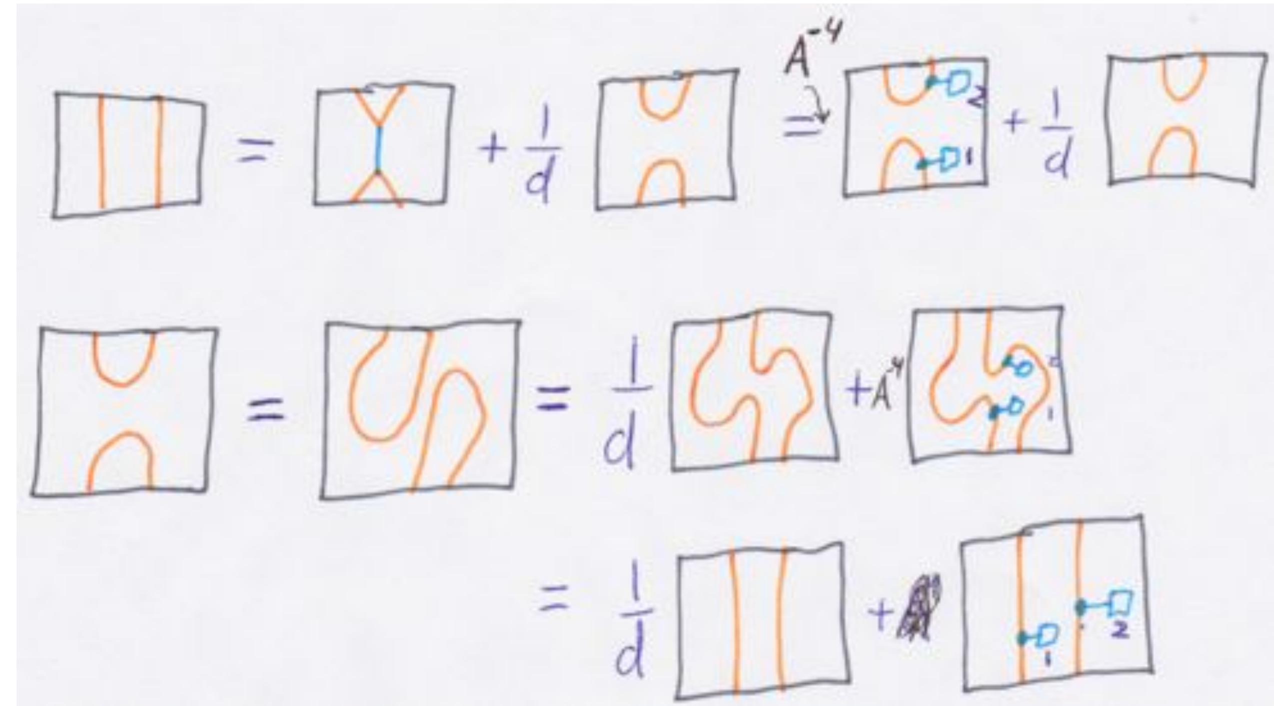


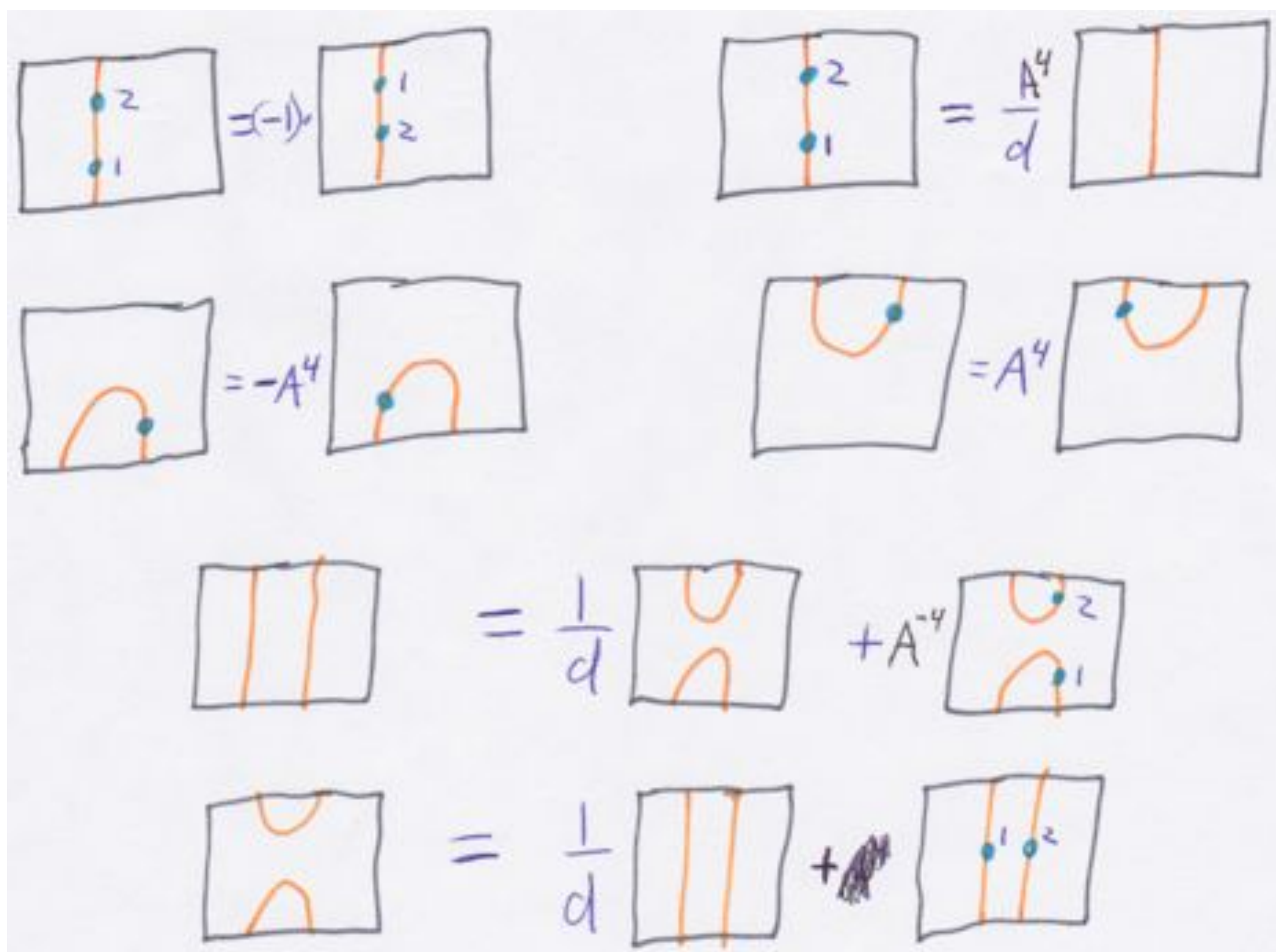


  - To define this flat line bundle, we must equip W with a spin structure.

- We can now shrink the third dimension and this on this as a 2D system – a super pivotal category (super 2-category).







- This category has only one non-trivial simple object, β (the image of σ in the quotient).
- But β is not so simple! It's endomorphism algebra is $Q \cong cl(1) \cong \mathbb{C}^{1/1}$.

Commuting projection hamiltonian

- We can follow the standard procedure and construct a commuting projection hamiltonian for this super TQFT.
- There is one new feature in the super case. We must replace

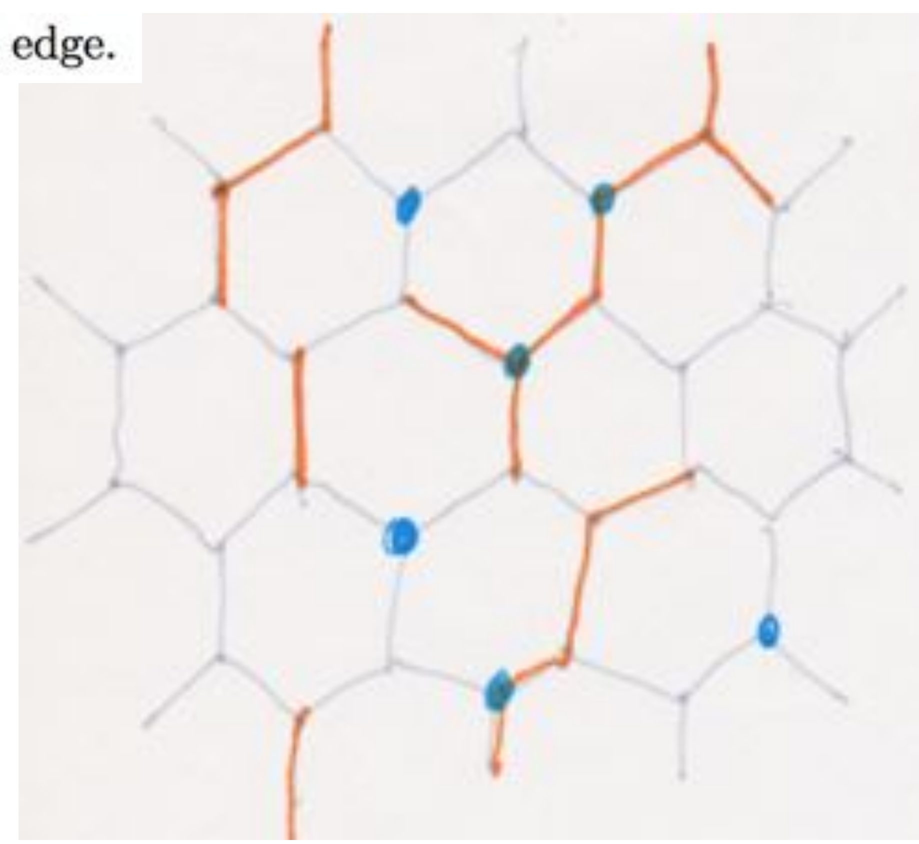
$$V_{abcd} \cong \bigoplus_x V_{abx} \otimes_{\mathbb{C}} V_{xcd}$$

with

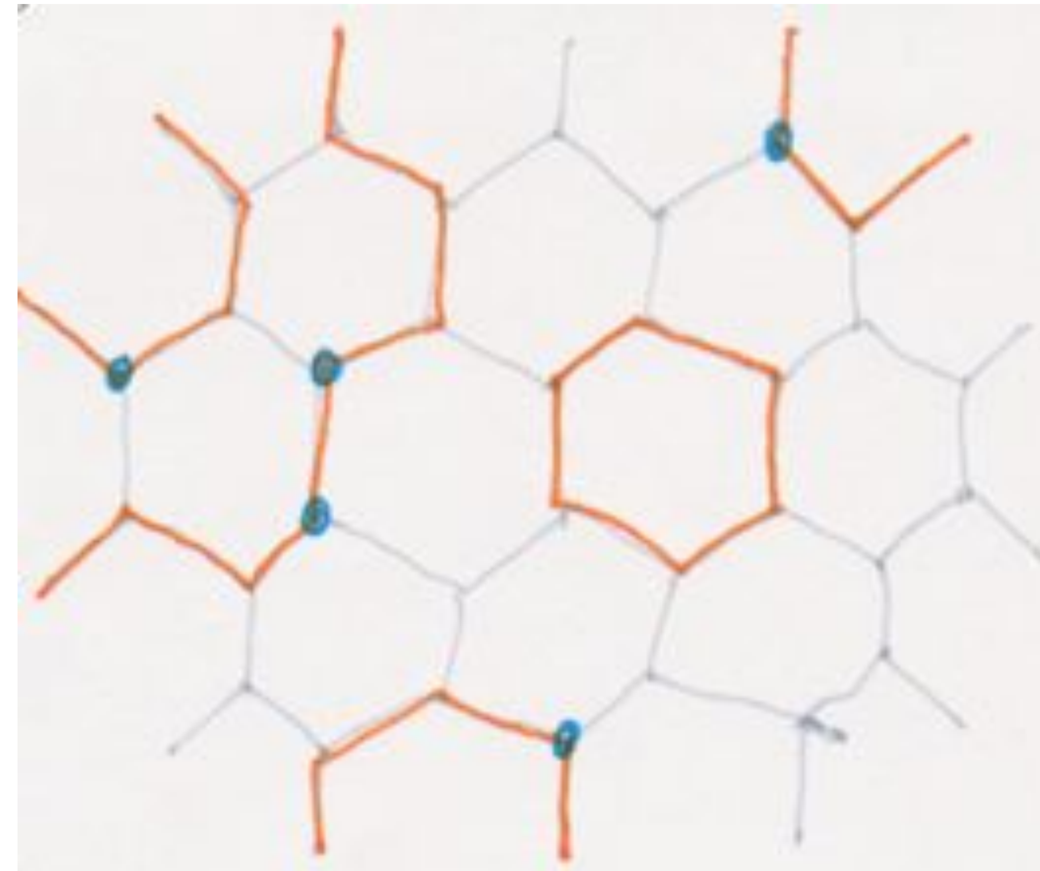
$$V_{abcd} \cong \bigoplus_x V_{abx} \otimes_{\mathrm{End}(x)} V_{xcd},$$

and corresponding changes must be made for $6j$-symbols, pentagon identities, etc.

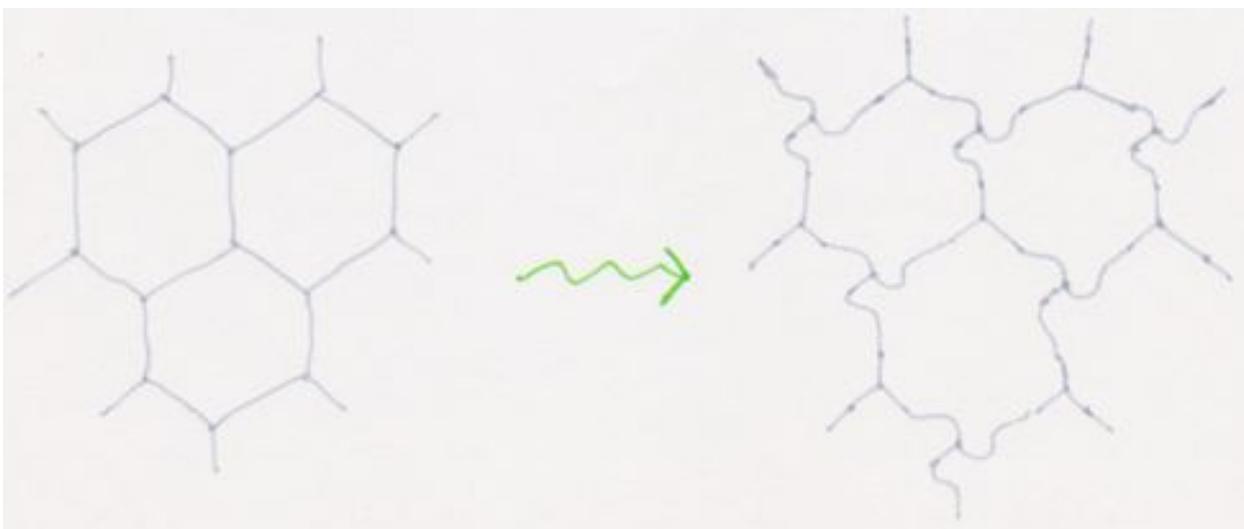
- The Hamiltonian has a spin at each vertex and a spin at each edge.

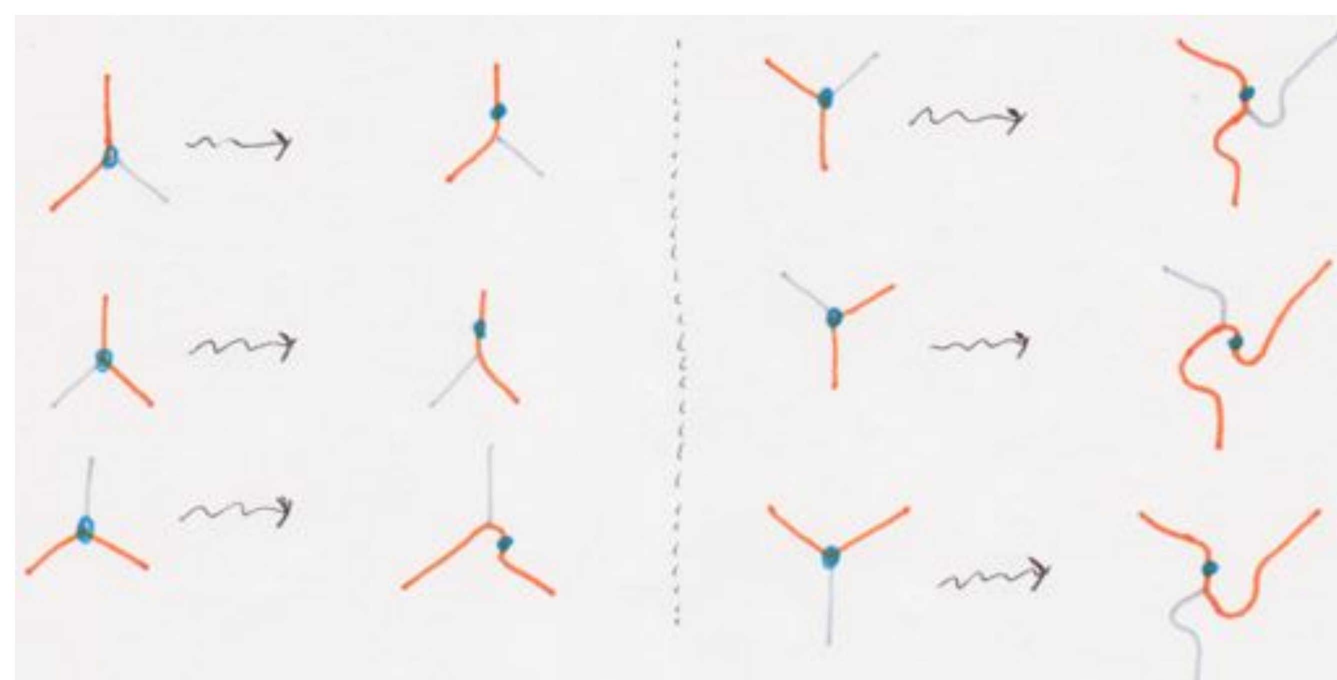


- There are (boring, bookkeeping) vertex terms in the hamiltonian which enforce β-parity and make sure the dots only appear on β-arcs.

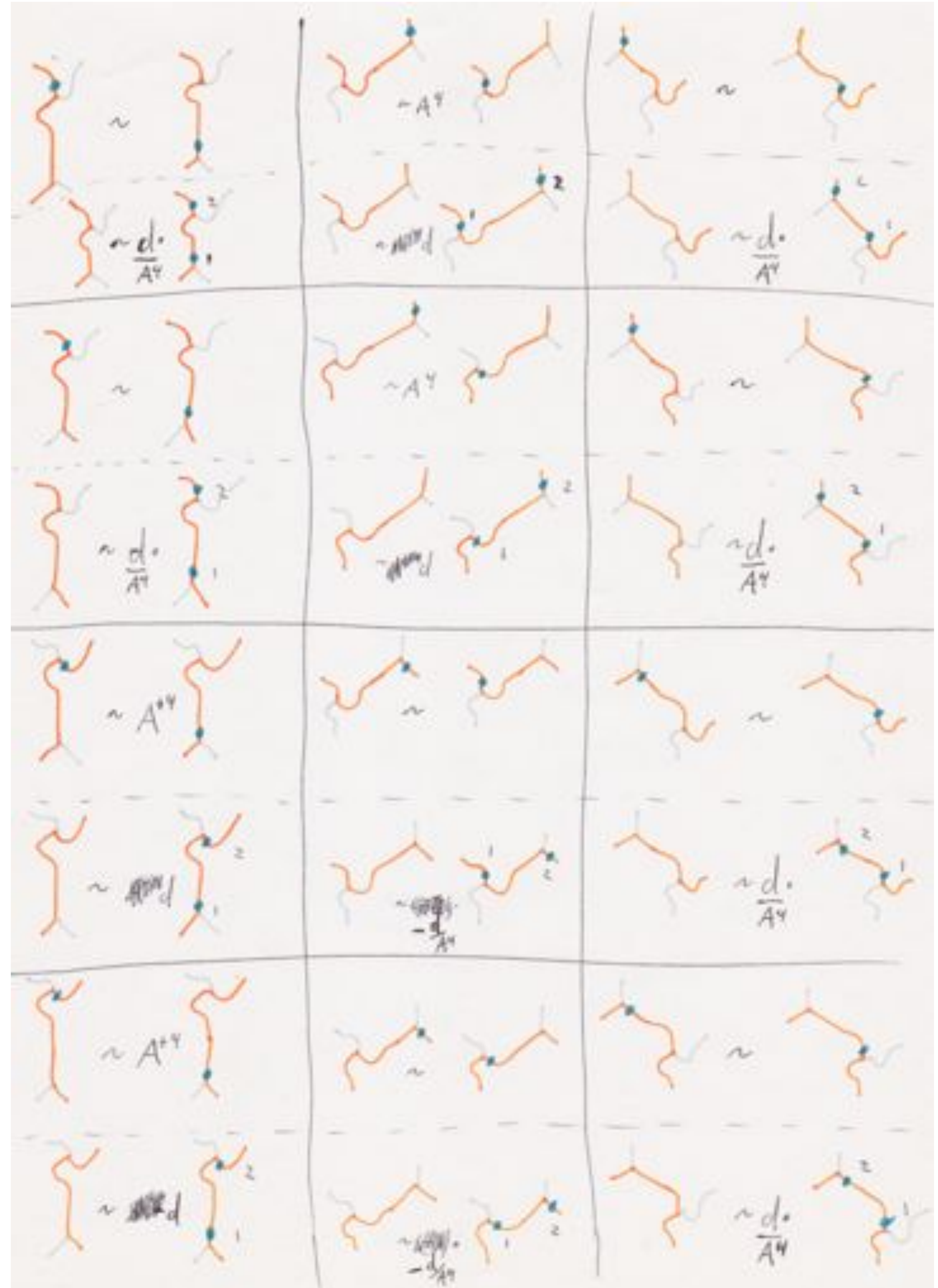


Before proceeding we must be more careful about framings





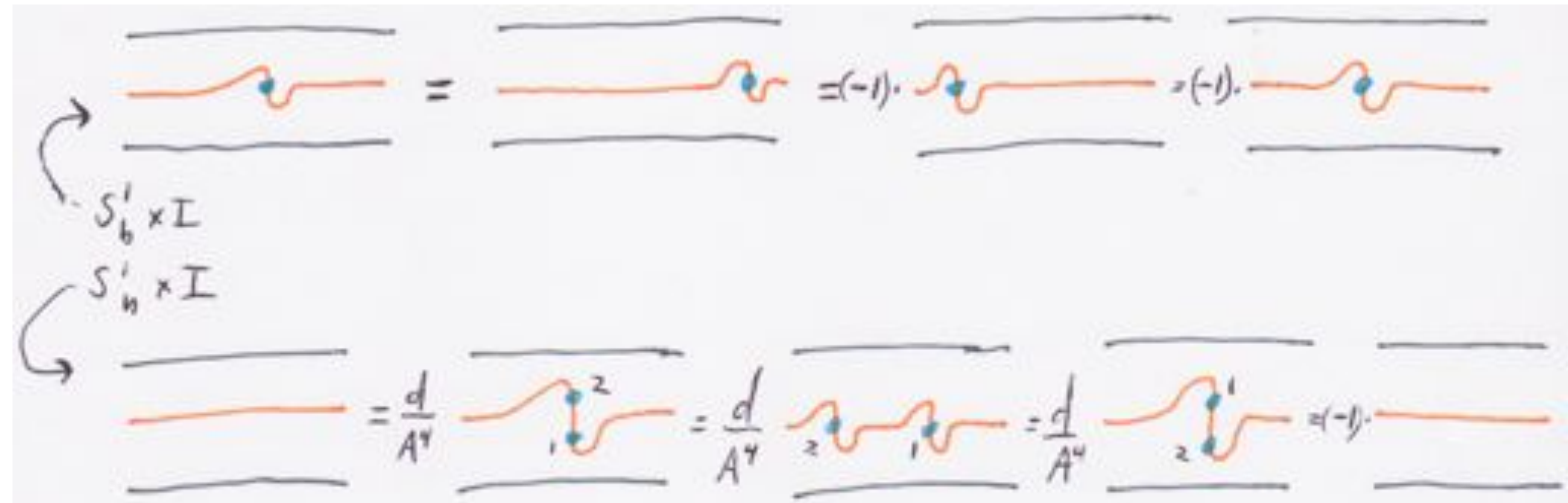
- The new feature of the hamiltonian are the edge terms which allow the dots to move along the β-arcs. These terms implement a tensor product over $\mathrm{End}(\beta)$. We must keep careful track of framings and phase factors.



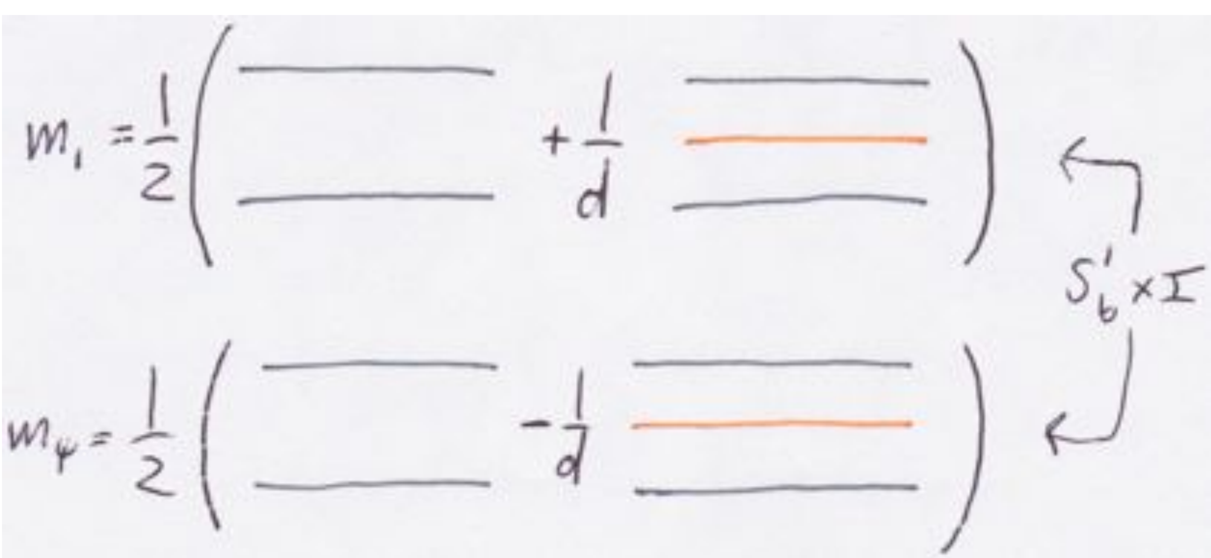
The 2-cell (placket) terms are similar to the usual Levin-Wen version. The main complication is that we must keep careful track of rotations and orderings of 3-valent vertices

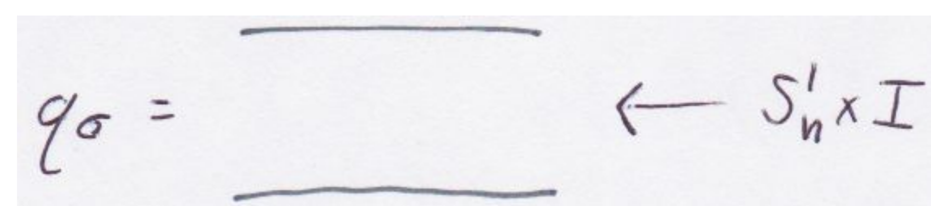
What are the anyons in this system?

- Recall that anyon types biject with irreducible representations of $A(S^1)$, but with spin structures there are two circles S^1_b (non-vortex) and S^1_n (vortex).



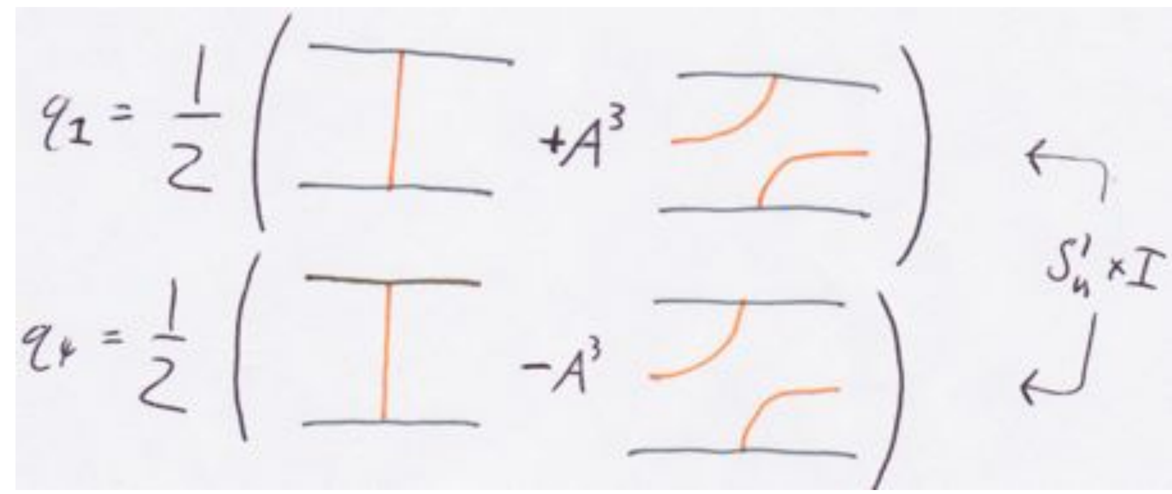
- $A(S^1_b; \emptyset, \emptyset) \cong \mathbb{C}^{2/0}$, so we get two M-type (ordinary) anyons.
- $A(S^1_n; \emptyset, \emptyset) \cong \mathbb{C}^{1/1}$, so we a single Q-type anyon.



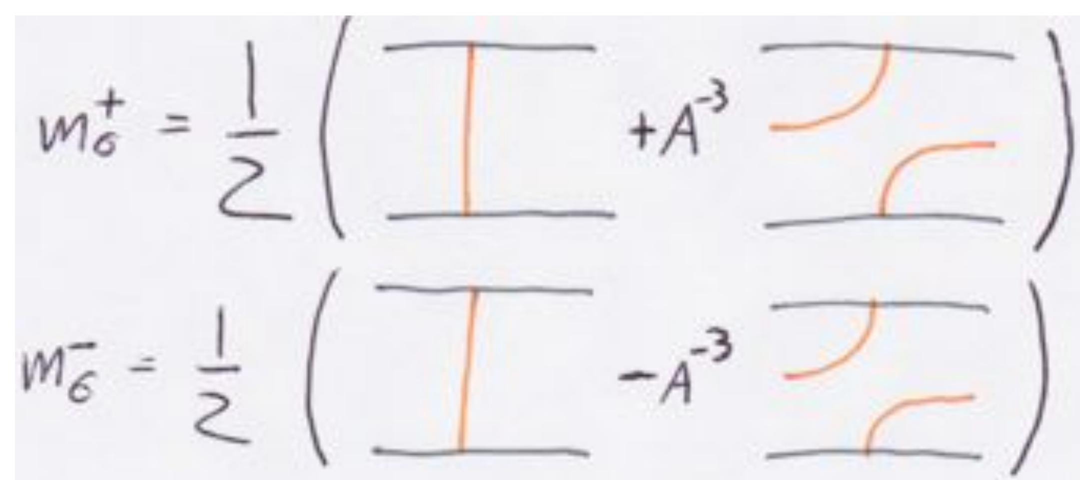


- $A(S^1_b;\beta,\beta) \cong A(S^1_n;\beta,\beta) \cong \mathbb{C}^{2/2}$, so as an algebra this is either $Q \oplus Q$ or $\mathrm{End}(\mathbb{C}^{1/1})$.

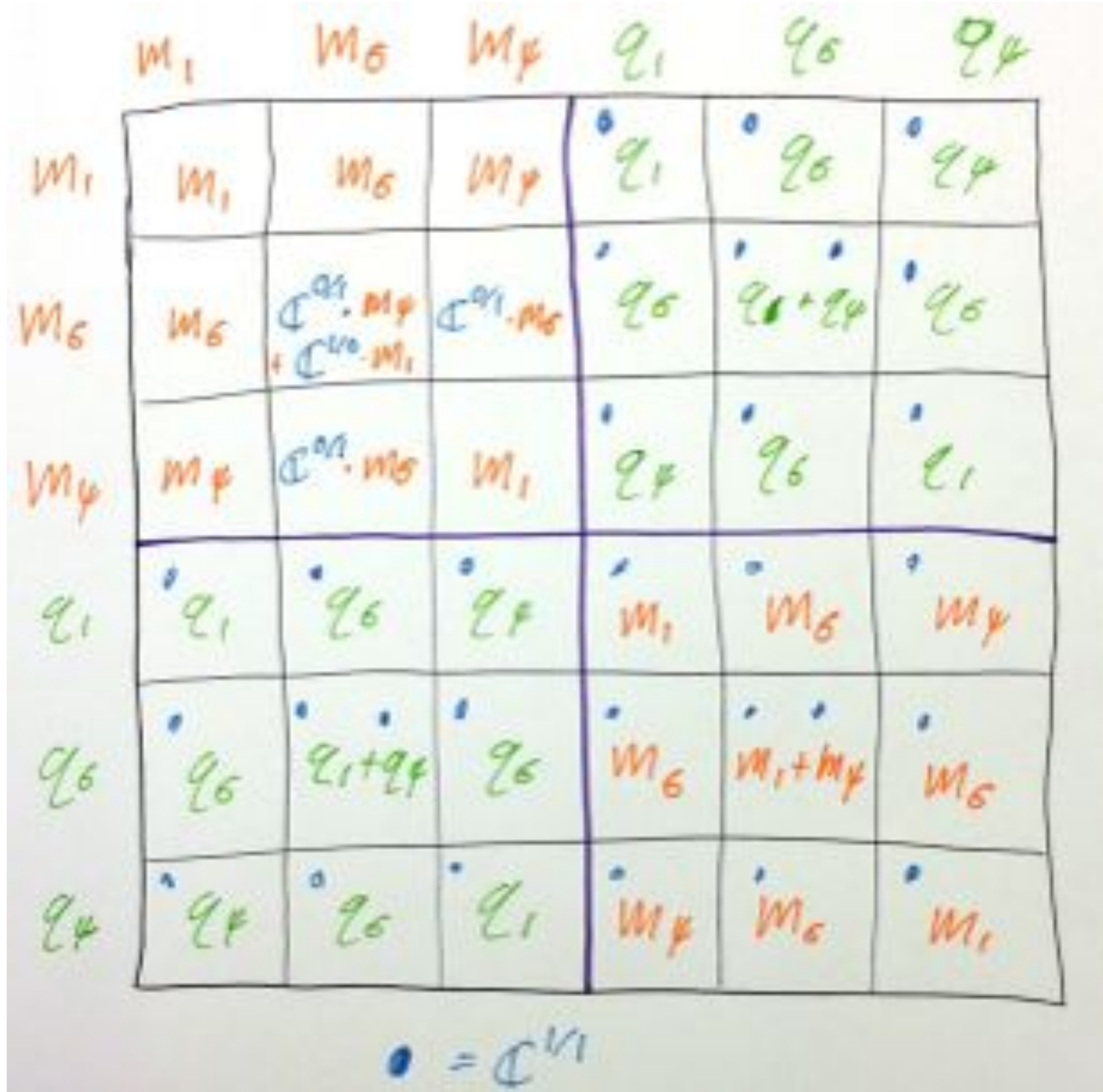
- For S^1_n, it's $Q \oplus Q$, and we get two more Q-type anyons.



- For S^1_b, it's $\mathrm{End}(\mathbb{C}^{1/1})$, and we we get a single equivalence class of M-type anyone (but two versions!). The two versions are **oddly** isomorphic.



Fusion rules:



	m_1	m_6	m_ψ	q_1	q_6	q_ψ
m_1	m_1	m_6	m_ψ	$\bullet q_1$	$\bullet q_6$	$\bullet q_\psi$
m_6	m_6	$\mathbb{C}^{0\vert 1}\cdot m_\psi + \mathbb{C}^{1\vert 0}\cdot m_1$	$\mathbb{C}^{0\vert 1}\cdot m_6$	$\bullet q_6$	$\bullet q_1 + \bullet q_\psi$	$\bullet q_6$
m_ψ	m_ψ	$\mathbb{C}^{0\vert 1}\cdot m_6$	m_1	$\bullet q_\psi$	$\bullet q_6$	$\bullet q_1$
q_1	$\bullet q_1$	$\bullet q_6$	$\bullet q_\psi$	$\bullet m_1$	$\bullet m_6$	$\bullet m_\psi$
q_6	$\bullet q_6$	$\bullet q_1 + \bullet q_\psi$	$\bullet q_6$	$\bullet m_6$	$\bullet m_1 + \bullet m_\psi$	$\bullet m_6$
q_ψ	$\bullet q_\psi$	$\bullet q_6$	$\bullet q_1$	$\bullet m_\psi$	$\bullet m_6$	$\bullet m_1$

$\bullet = \mathbb{C}^{1|1}$

One can construct many different super TQFTs in this way, including a super version of the Fibonacci theory

- Note that if we restrict to non-vortices, the fusion rules (as well as twists and braidings, up to signs) are the same as in the original Ising TQFT. But this version has a purely 2D commuting projection hamiltonian