Premodular TQFTs

Some parts of this are not yet published.
Other parts can be found at http://canyon23.net/math/tc.pdf (2005)
and arXiv:1009.5025 (2010).

These slides available at http://canyon23.net/math/talks/

Let's recall some results/constructions from the '80s and '90s

- Witten Chern-Simons TQFT (1988), a.k.a. Reshetikhin-Turaev invariants
- Crane-Yetter-Kauffman state sum, TQFT (1993)
- Turaev "shadow" state sum (1993(?))
- Homotopy TQFTs derived from quantum groups (199x)
- Spin TQFTs derived from quantum groups (199x)

We will see that all of the above are all aspects a single 3+1-dimensional TQFT

Could be called either the Crane-Yetter-Kauffman TQFT or the premodular TQFT

Outline

1. Review TQFT framework
2. Premodular TQFT, first look
3. Modular special case
4. Bimodules
5. Morita equivalence
6. The Fourier bimodule applied to premodular TQFTs

Very quick review of the TQFT framework I'll be using

- For a physicist, the basic ingredients of a TQFT are (1) fields $\mathcal{F}$, (2) an action functional $S(x)$, and (3) the path integral

$$Z(W^{n+1}) := \int_{x \in \mathcal{F}(W)} e^{iS(x)}$$

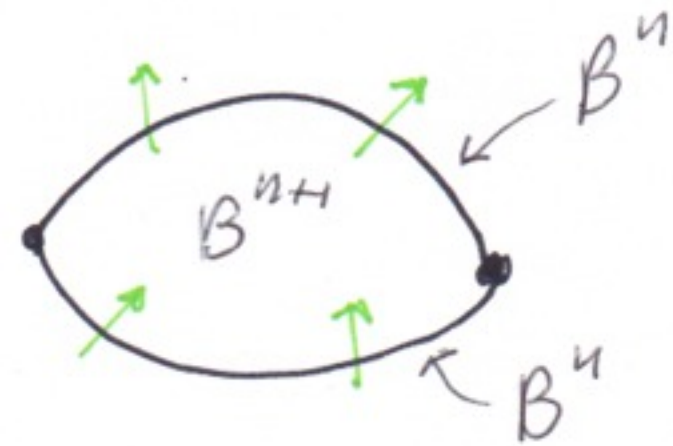
- From this starting point one builds an elaborate structure, including a functor defined on the cobordism $(n+1)$-category and asymptotic expansions

- It is difficult to make the the path integral rigorous, so as mathematicians we seek an alternative starting point

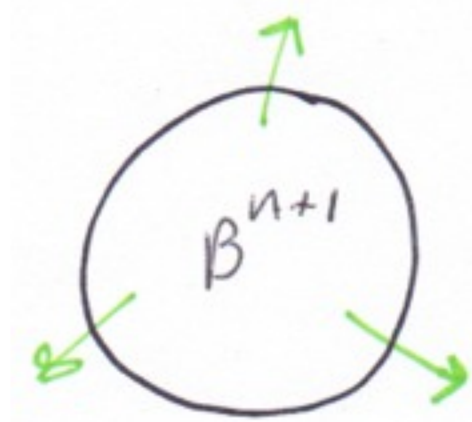
- Atiyah-Segal idea: use the functor on cobordism category as the starting point; define a TQFT to be such a functor

- But this leaves out parts of the physics picture that can easily be made rigorous: fields
- So we want an alternative starting point that includes fields

- We're still afraid of the path integral, so we need a substitute for that
- It turns out that if we know $Z(B^{n+1})$, the path integral of the $(n+1)$-ball thought of as a bordism from B^n to B^n, then we can reconstruct the 0- through n-dimensional parts of the TQFT using just algebra and cut-and-paste topology



- If, in addition, we know $Z(B^{n+1})$, where this time we think of B^{n+1} as bordism from the empty n-manifold to the sphere ∂B^{n+1}, then we can compute (combinatorially) the path integral for any $(n+1)$-manifold

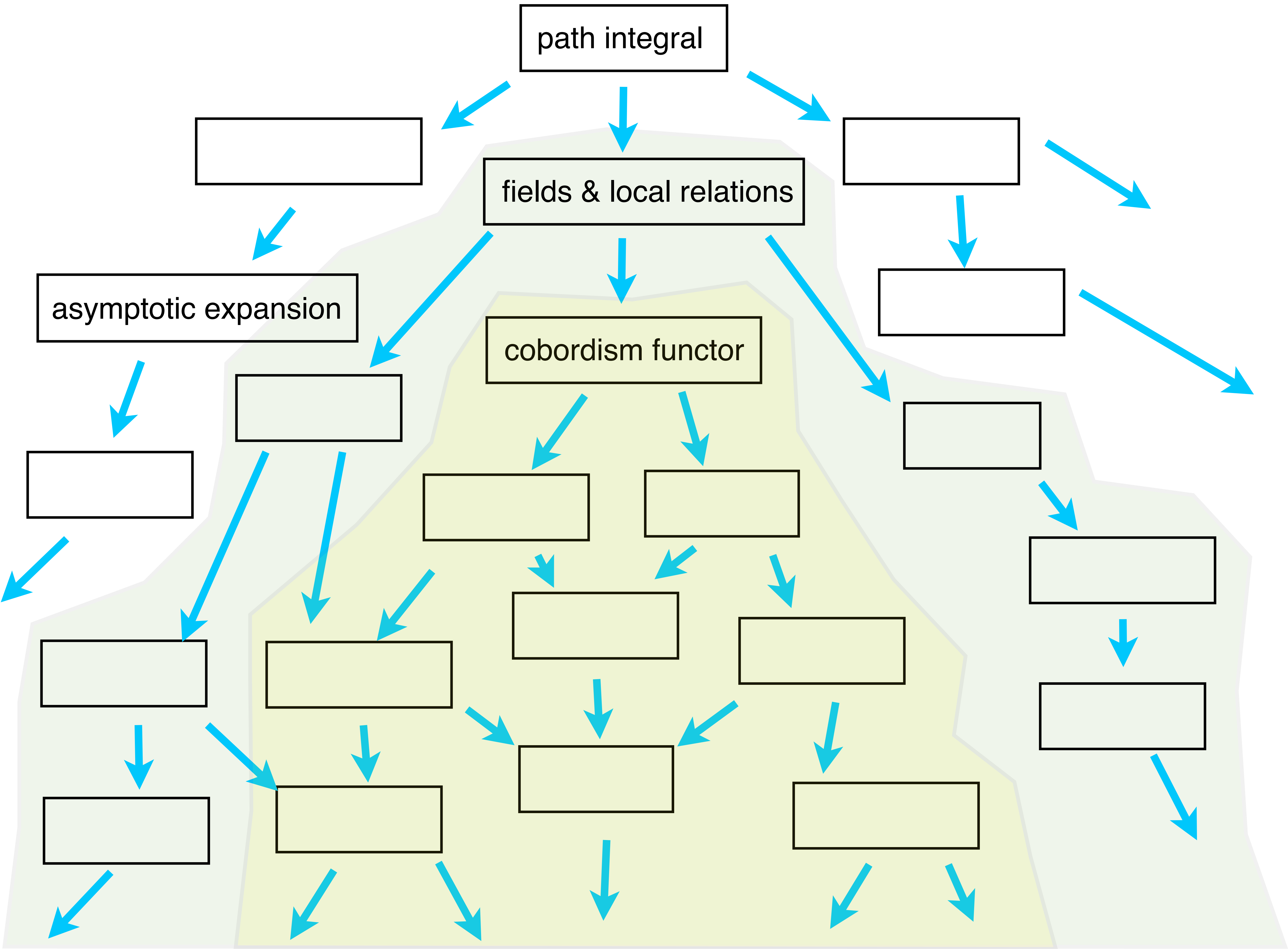


$(n+\epsilon)$-dimensional TQFT

$(n+1)$-dimensional TQFT

- In conclusion, we define a TQFT to be the data
  1. fields for 0- through n-dimensional manifolds
  2. local projections $Z(B^{n+1})$ (a different one for each field on ∂B^n)
  3. (optionally) inner product or trace $Z(B^{n+1})$

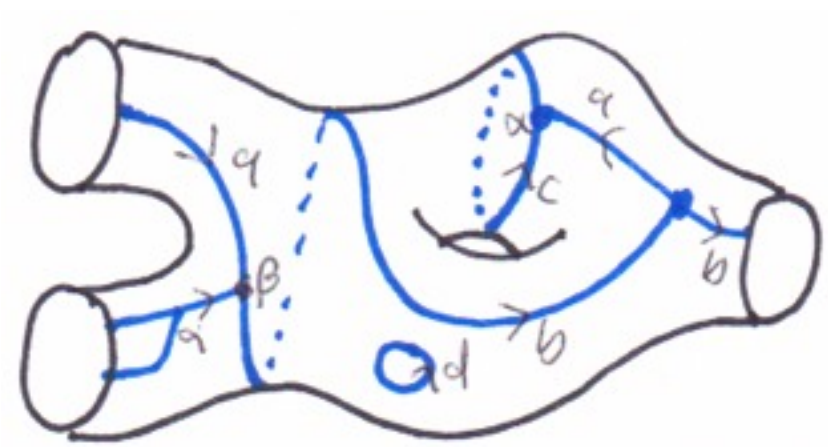
  This data is required to satisfy a relatively simple list of axioms (I won't give the full details here)

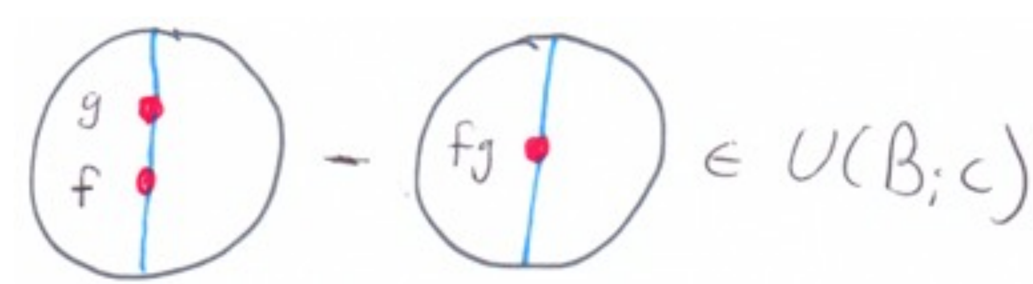

path integral
fields & local relations
asymptotic expansion
cobordism functor

- It turns out to be more convenient to replace local projections with their predual, local relations. A finite linear combination of fields is zero in the local relation if it is in the kernel of the local projection.
- Notation:
$$U(B;c) := \ker\left(Z(B\times I) : \mathbb{C}^{\mathcal{F}(B;c)} \to \mathbb{C}^{\mathcal{F}(B;c)}\right)$$
where B is an n-ball and $c \in \mathcal{F}(\partial B)$ is a boundary condition

- Example 1: $\mathcal{F}(X^k) := \{\text{continuous maps } X \to BG\}$, for $0 \le k \le n$. Local relation is homotopy rel boundary; $U(B;c)$ is generated by linear combinations $f_0 - f_1$, where $f_i : B \to BG$ and f_0 and f_1 are homotopic rel boundary. (Alternatively, use homotopy twisted by an $(n{+}1)$-cocycle on BG.)

- Example 2: $\mathcal{F}(X^k) := \{C\text{-string diagrams on } X\}$, where C is an appropriate type of n-category (e.g. strict pivotal 2-category). The local relations are given by the kernel of evaluation maps:
$$U(B;c) := \ker\left(\text{eval} : \mathcal{F}(B;c) \to \text{mor}^n(C)\right)$$





Rough sketch of axioms

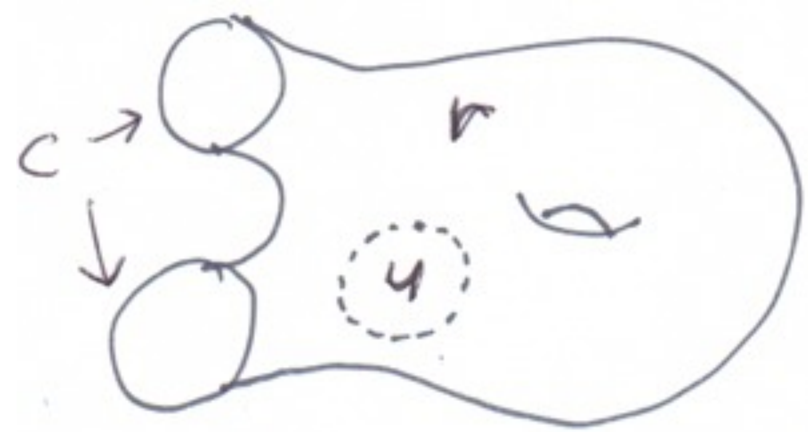
- Axioms for fields: behave nicely with respect to (1) restriction to boundary, (2) gluing, and (3) fibrations.

- Axioms for local relations: (1) at least as strong as isotopy (the "T" in TQFT), and (2) compatible with sub-covers.

- Axioms for $Z(B^{n+1})$: induces non-degenerate inner products on Hilbert spaces.

Constructing the TQFT from the input data

- For M an n-manifold and $c \in \mathcal{F}(\partial M)$, define a vector space



$$A(M;c) := \mathbb{C}[\mathcal{F}(M;c)]/U(M;c),$$

where the null fields $U(M;c)$ are generated by

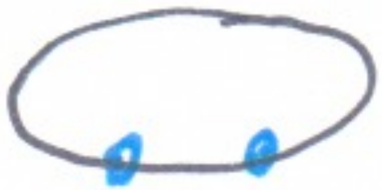
$$\{u \bullet r \mid B \subset M, u \in U(B), r \in \mathcal{F}(M \setminus B)\}$$

- $A(M;c)$ is the pre-dual to the Hilbert space

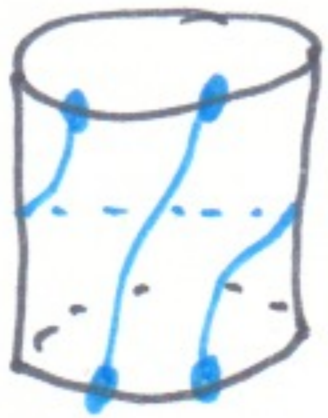
$$Z(M;c) := A(M;c)^*.$$

$Z(M;c)$ consists of functions on fields which are invariant under local relations. Equivalently, it is the intersection of all the local projections on M.

- For Y an $(n-1)$-manifold and $c \in \mathcal{F}(\partial Y)$, define $A(Y;c)$ to be the 1-category



$$\text{objects} : \mathcal{F}(Y;c)$$
$$\text{morphisms } a \to b : A(Y \times I; a, b)$$
$$\text{composition} : \text{stacking}$$



- Define $Z(Y;c)$ to be the representation 1-category

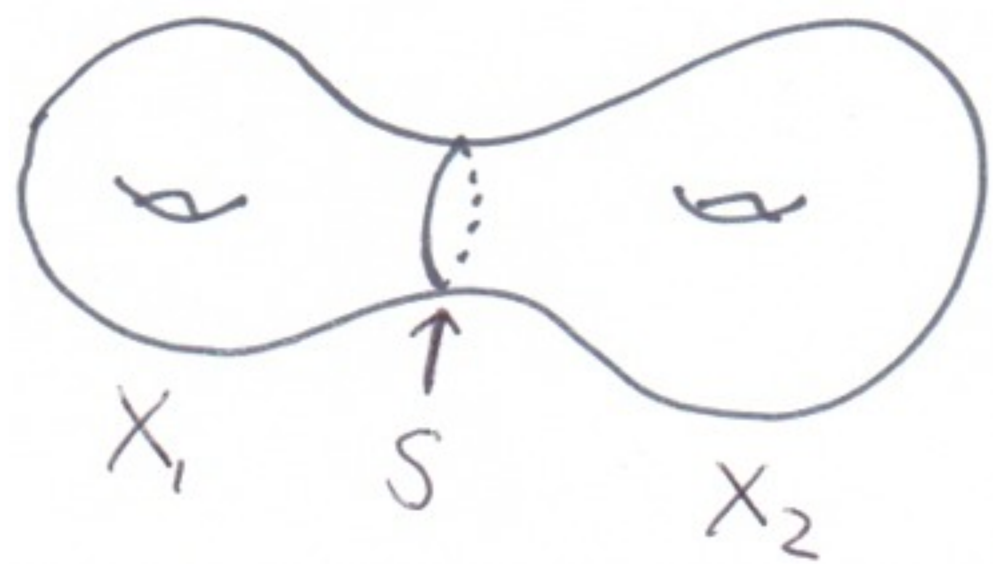
$$Z(Y;c) := \mathrm{Rep}(A(Y;c))$$

- In general, for X and $(n-k)$-manifold define $A(X)$ to be the k-category

$$j\text{-morphisms} : \mathcal{F}(Y \times B^j) \qquad j < k$$
$$k\text{-morphisms} : A(Y \times B^j)$$

(Omitting boundary conditions from the notation.) And define

$$Z(X) := \mathrm{Rep}(A(X))$$

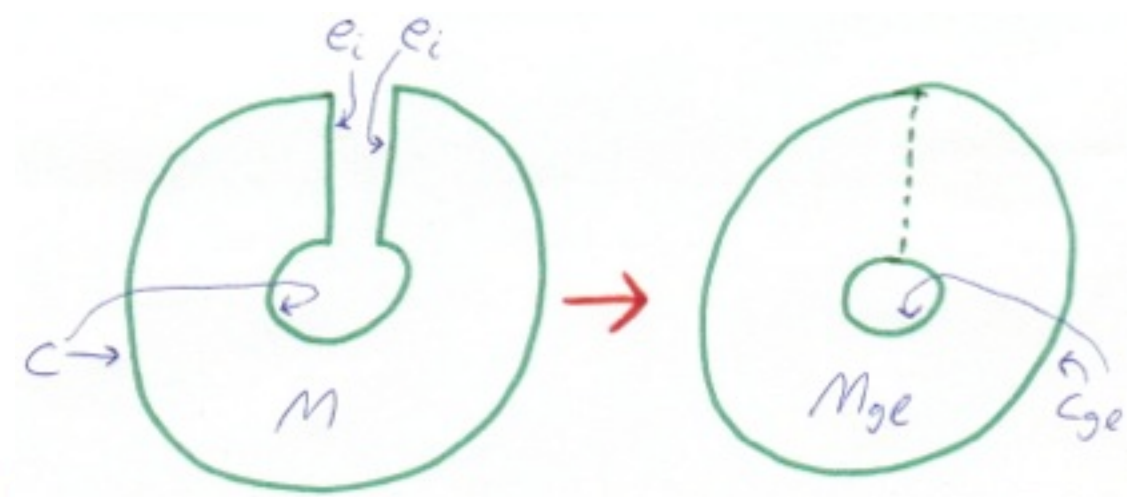


- The above definitions obey gluing formula

$$A(X_1 \cup_S X_2) \cong A(X_1) \otimes_{A(S)} A(X_2).$$

$A(X_i) := \{A(X_i; \cdot)\}$ is a family of k-categories (depending on the boundary condition) affording a representation of the $(k+1)$-category $A(S)$.

From (n+ε)-dimensional to (n+1)-dimensional



What we want from a path integral:

- For all M^{n+1}, $Z(M) \in Z(\partial M)$, i.e.

$$Z(M) : A(\partial M) \to \mathbb{C}$$

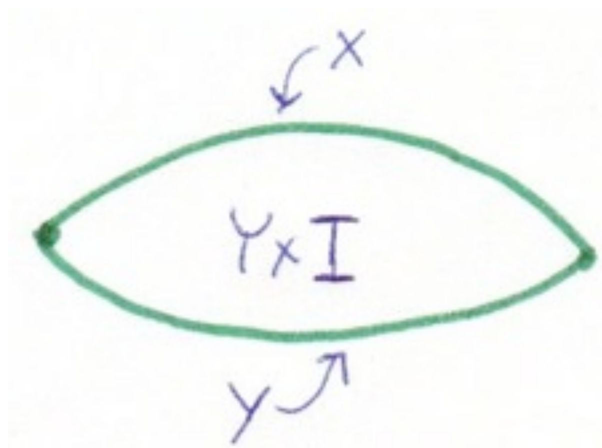
- satisfying the gluing formula

$$Z(M_{\text{gl}})(c_{\text{gl}}) = \sum_i Z(M)(e_i \bullet e_i \bullet c) \frac{1}{\langle e_i, e_i \rangle},$$

  where e_i runs through an *orthogonal* basis of $A(Y; \partial c)$

- and where the (non-degenerate) inner products of $A(Y^n; b)$ are related to the path integral via

$$\langle x, y \rangle = Z(Y \times I)(x \bullet y)$$



Theorem. Suppose

1. there exists $z \in Z(S^n)$ such that the induced inner product $A(B^n; c) \otimes A(B^n; c) \to \mathbb{C}$ given by $a \otimes b \mapsto z(a \bullet b)$ is positive definite for all $c \in \mathcal{C}(S^{n-1})$; and

2. $\dim A(Y^n; c) < \infty$ for all n-manifolds Y and all $c \in \mathcal{C}(\partial Y)$.

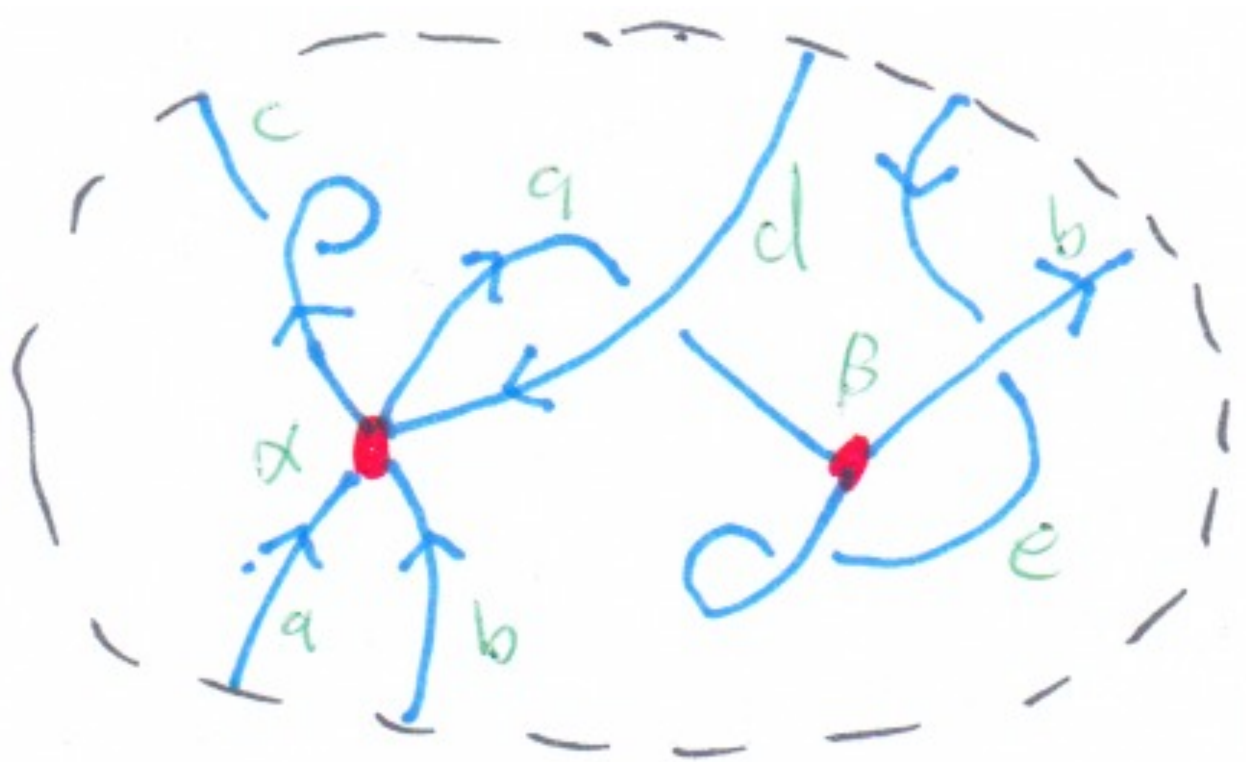
Then there exists a unique path integral $Z(M^{n+1}) \in Z(\partial M)$ (for all $n+1$-manifolds M) satisfying the the above conditions and such that $Z(B^{n+1}) = z$.

Input (fields)	State sum	Commuting projection hamiltonian
maps to BG	Dijkgraaf-Witten finite group state sum	Kitaev finite group model
Rep(G)	(no name?)	(no name?)
semisimple 1-category	Verlinde type sum (Euler characteristic theory)	1-dimensional ferromagnet
pivotal 2-category	Turaev-Viro state sum	Levin-Wen model
premodular category (3-category)	Crane-Yetter sum or Reshetikhin-Turaev surgery formula or Turaev “shadow” sum	“WW” model

TQFT built out of a premodular category

Premodular category:

- 3-category ($\Rightarrow$ 3-dimensional string diagrams, (3+ϵ)-dimensional TQFT)
- trivial 0- and 1-morphisms ($\Rightarrow$ string diagrams look like ribbon graphs, not like foams)
- 3-pivotal / disk-like / strong duality ($\Rightarrow$ can rotate the string diagrams)
- semisimple; finitely many equivalence classes of simple objects ($\Rightarrow$ finite dimensional Hilbert spaces, path integral defined for any 4-manifold, (3+1)-dimensional TQFT)



Applying the above general recipe, we build of the premodular TQFT based on the premodular category C as follows:

- $A(M^3)$ is the skein module based on C-ribbon graphs modulo the usual relations; $Z(M)$ is functions on skeins invariant under local relations
- $A(Y^2)$ is a 1-category whose objects are configurations of ribbon end points on Y and whose morphisms are ribbon graphs in $Y \times I$
- $A(X^1)$ is a 2-category whose 0-, 1- and 2-morphisms are diagrams in X, $X \times B^1$ and $X \times B^2$
- $A(pt)$ is equivalent to the 3-category C that we started with; $Z(pt) = \mathrm{Rep}(C)$

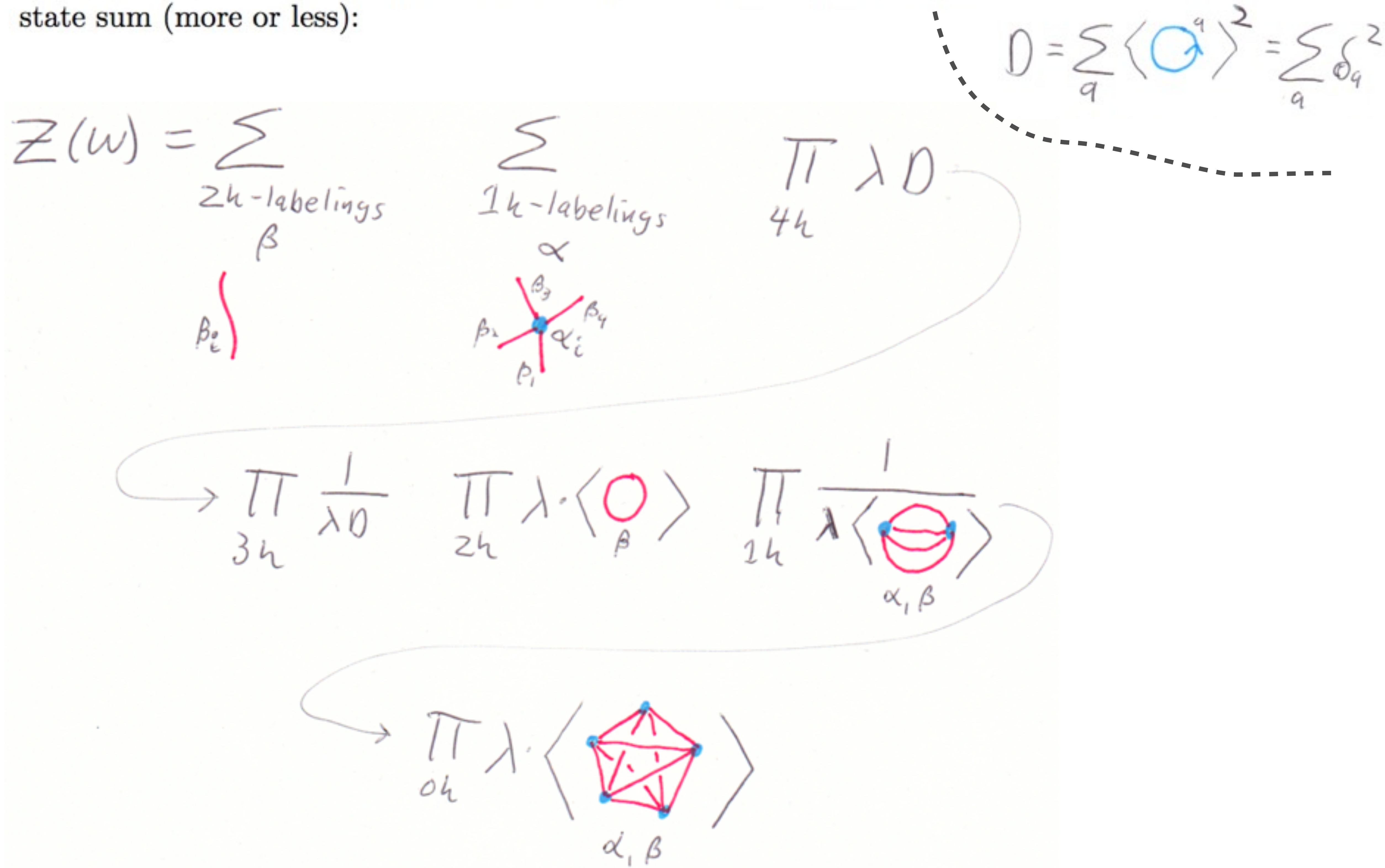
- $A(S^3)$ is a 1-dimensional vector space; any diagram is a scalar multiple of the empty diagram
- So $Z(B^4) : A(S^3) \to \mathbb{C}$ is determined by it value on the empty diagram

$$Z(B^4)(\emptyset) = \lambda$$

We can take λ to be any non-zero real number

Applying the path integral theorem, we can derive a state sum computing $Z(W^4)$ for any style of decomposing W into a union of 4-balls:

- For a generic handle decomposition of W^4 (dual to a triangulation), we get the Crane-Yetter state sum (more or less):

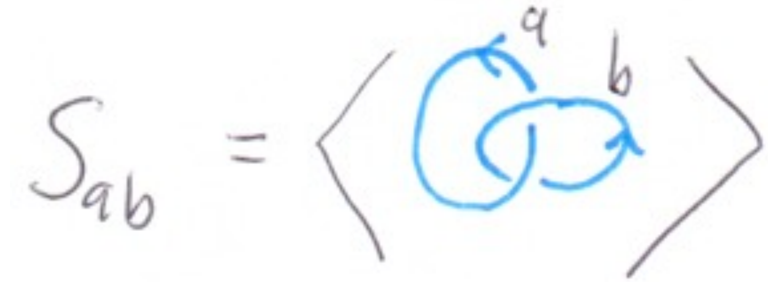


- For B^4 union 2-handles along a link $L \subset \partial B^4$, we get the Reshetikhin-Turaev surgery formula:

$$Z(W)(\Gamma) = \sum_{\beta \in \text{labelings of } L} \lambda \cdot J(\Gamma \cup L_\beta) \cdot \prod_{2h} \lambda \cdot \langle O_\beta \rangle$$

- If W is a thickened neighborhood of a 2-complex, we get the Turaev "shadow" state sum

Modular case



Recall that C is modular if

$$\det[S_{ab}] \neq 0 \quad \Longleftrightarrow \quad A(S^2) \cong \text{the trivial 1-category}$$

At first the modular case looks very boring:

- $A(M^3)$ is 1-dimensional for any closed 3-manifold
- $A(Y^2)$ is Morita trivial (full matrix category) for any closed 2-manifold
- $Z(W^4) = \tau^{\sigma(W)}$ for any closed 4-manifold W. Here τ is the (exponentiated) central charge, $\sigma(W)$ is the signature of W, and we are taking $\lambda^2 = 1/D$.

But for manifolds with non-empty boundary things are more interesting:

- Let W be a 4-manifold and c be a C-diagram (labeled ribbon graph) in ∂W. Then

$$Z(W)(c) = Z_{CS}(\partial W, \Gamma),$$

  where Z_{CS} denotes the Witten-Reshetikhin-Turaev Chern-Simons TQFT.

- Let M be a 3-manifold and c be a finite set of labeled framed points (ribbon end points) in ∂M. Then

$$Z(M; c) \cong Z_{CS}(\partial M, c).$$

- Let Y be a 2-manifold with m boundary components. Then

$$Z(Y; \emptyset) \cong Z_{CS}(\partial Y) \cong C \times \cdots \times C \quad (m \text{ copies of } C)$$

  (On the RHS we are forgetfully treating C as a 1-category.)

Turning this around, we can define the (2+1)-dimensional, not-quite-fully-extended Chern-Simons TQFT via

$$Z_{CS}(X, c) := Z(\partial^{-1} X; c).$$

Here

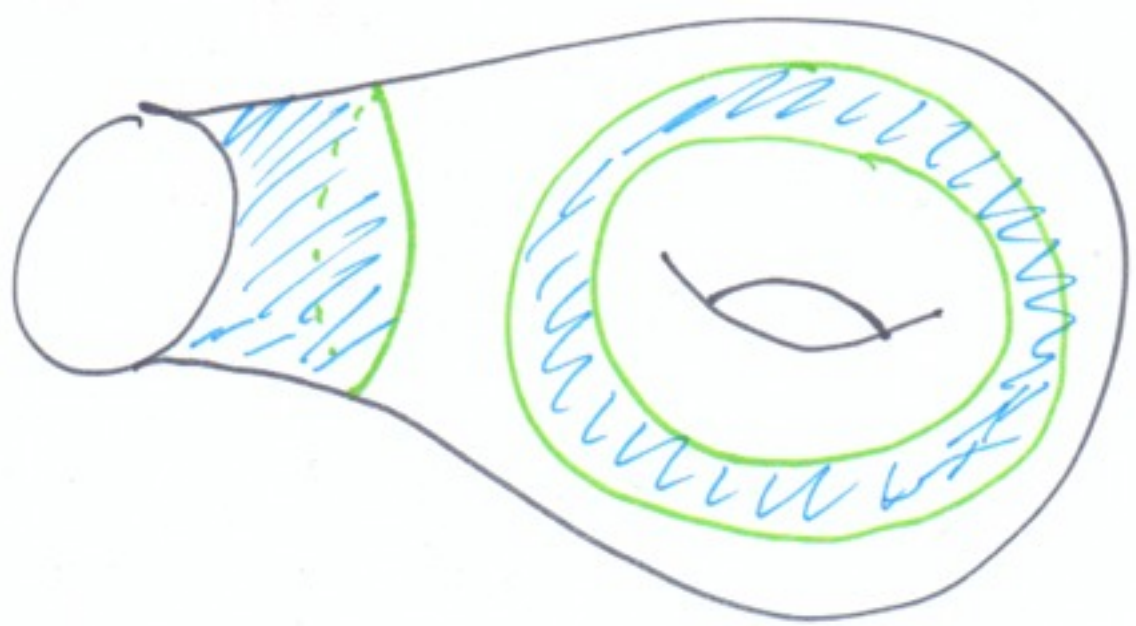
- X has dimension 1, 2 or 3. (Can't define for points since points don't bound.)
- X must be equipped with enough extra structure to pick out $\partial^{-1} X$ up to Z-equivalence. (e.g. equipped with null-bordism, p_1-structure, or "2-framing")
- On the LHS, c is a decoration of X (e.g. embedded ribbon graph or "Wilson loop"); on the RHS c is a boundary condition for $\partial^{-1} X$.

- Note that the (3+1)-dimensional TQFT based on a modular category is a fully extended, "4-3-2-1-0" theory, but the Chern-Simons (2+1)-dimensional TQFT is only a "3-2-1" theory.

- Note that in the above arguments we have made heavy use of the preduals $A(\cdots)$ and concrete boundary conditions. These are not immediately available in the Atiyah-Segal axiom approach to TQFTs.

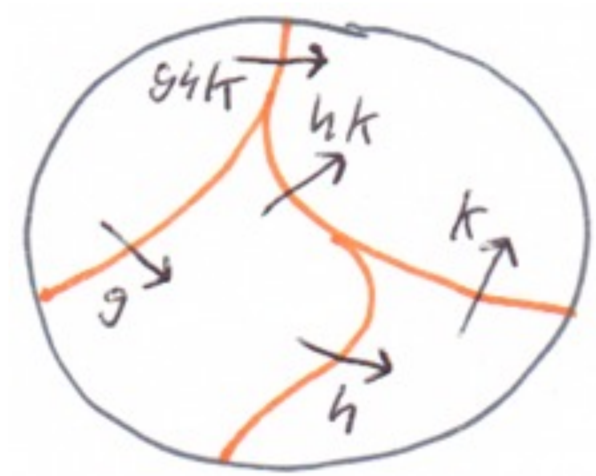
Bimodules

- We can think of an n-category as the local part of an $(n+\epsilon)$-dimensional TQFT. Take the above definition of fields and local relations and restrict to the case where all manifolds are homeomorphic to balls. (This is made precise in arXiv:1009.5025.)
- In the TQFT definition, we can replace ordinary, undecorated manifolds with manifolds equipped with a labeled stratification. This leads to an axiomatization of TQFTs with defects.

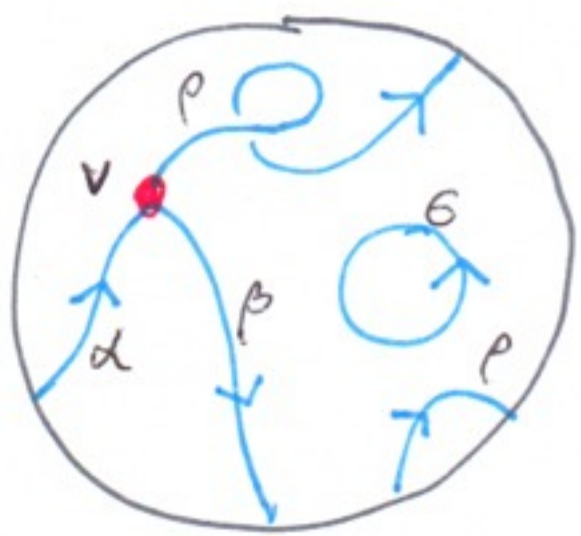


- We can think of a bimodule between two n-categories as the local part of a codimension-1 defect.
- In lieu of a detailed definition, here's an example of a bimodule...

- Let G be a finite group. For any m, we have the m-category $G_m := \pi_{\leq m}(BG)$. Equivalently we can define G_m to be the m-category of "G-foams" embedded in balls. This m-category corresponds to the untwisted $(m+1)$-dimensional Dijkgraaf-Witten theory based on G (fully extended version).

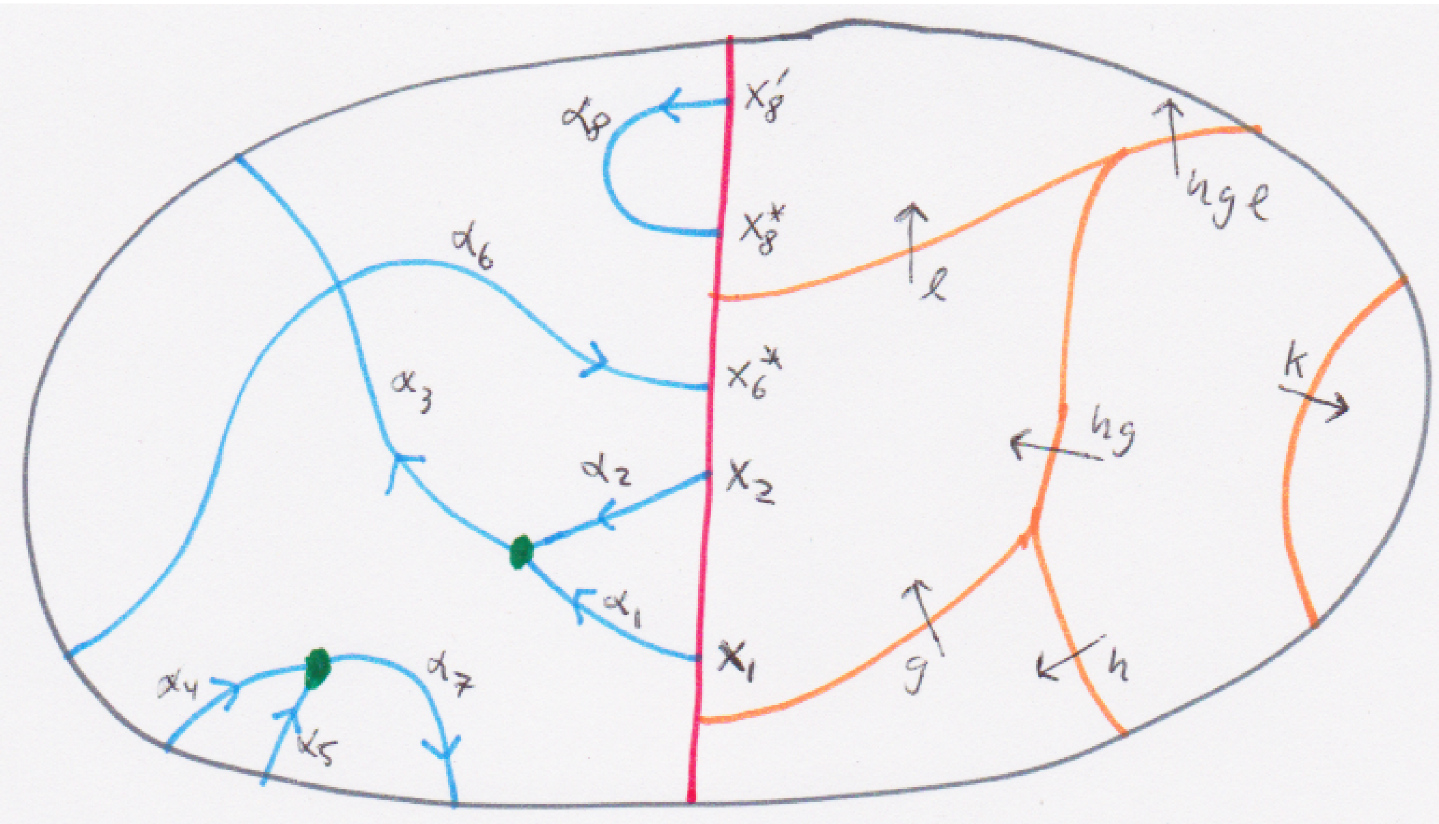


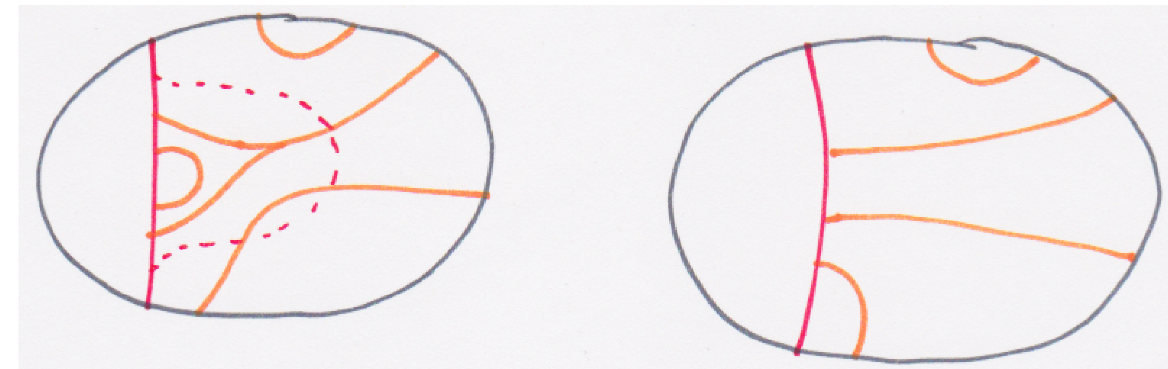
- Let $R = \mathrm{Rep}(G)$ be the symmetric monoidal category of representations of G. For any m, we have the m-category R_m of R-string diagrams embedded in m-balls.

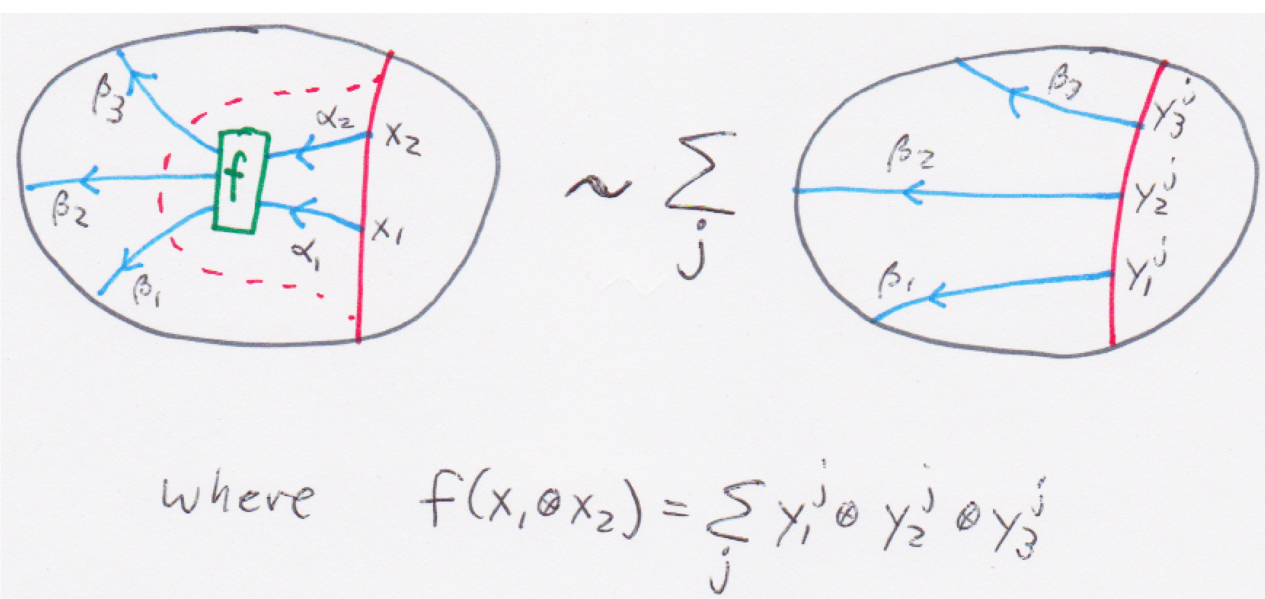


- Note that there are no interesting k-morphisms of G_m for $k > 1$, while there are no interesting k-morphisms of R_m for $k < m - 1$.

- We define a bimodule F_{m-1} between G_m and R_m as follows. (F stands for Fourier transform.)





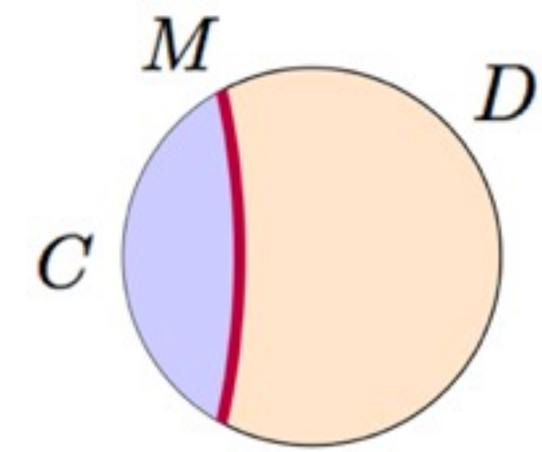


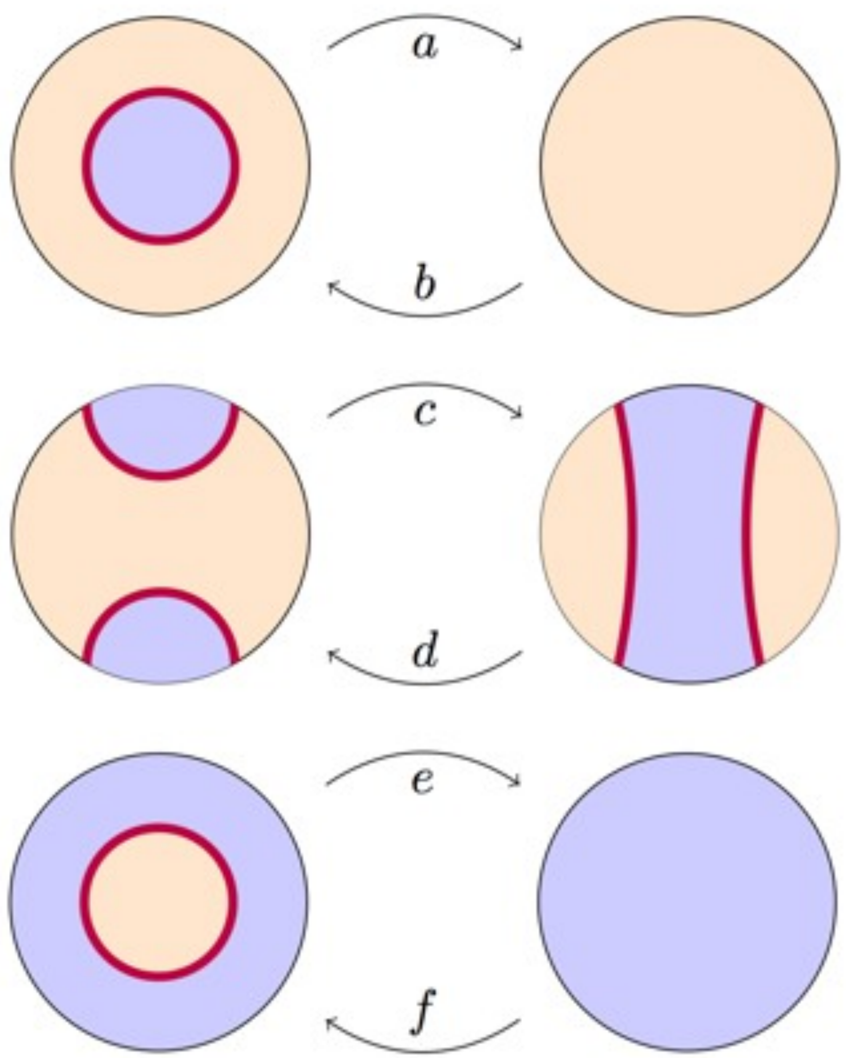
Morita equivalence for n-categories with duality

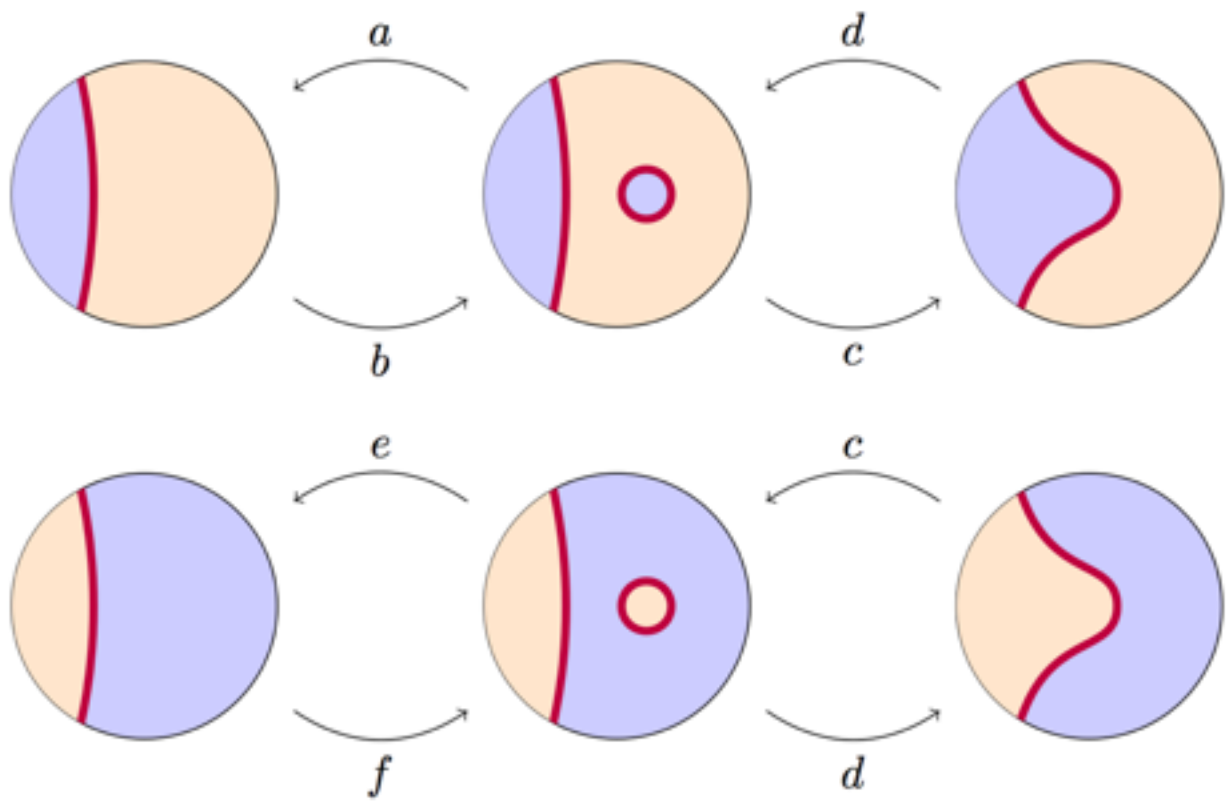
Definition. A Morita equivalence between n-categories C and D consists of

- a C-D bimodule M
- isomorphisms for each local "Morse move" on C-D-colored n-manifolds

This data is required to satisfy a relation for each possible Morse cancellation.







- Morita equivalent n-categories give rise to equivalent $(n+\epsilon)$-dimensional TQFTs. More specifically, for any closed $(n-k)$-manifold Y, $A_C(Y)$ and $A_D(Y)$ are Morita equivalent k-categories.

- Note that because of duality we specify much less data than in a typical Morita equivalence definition.

- If we equip the bimodule M with an inner product, we can extend the definition of Morita equivalence so that it yields equivalent $(n+1)$-dimensional TQFTs.

Theorem. The bimodule F_{m-1} is a Morita equivalence between G_m and R_m (for any m).

The proof boils down to the familiar fact that $\mathbb{C}[G] \cong \bigoplus \mathrm{End}_{\mathbb{C}}(\rho)$, both as commutaive algebras and $\mathbb{C}[G]$-$\mathbb{C}[G]$ bimodules.

Another Morita equivalence

- Let C be a modular category.
- We can think of the disk D^2 as a bordism between S^1 and the empty 1-manifold.
- We can view the collection of 1-categories $A_C(D^2)$ as a bimodule between the 2-category $A_C(S^1)$ and the trivial 2-category.

Theorem. The bimodule $A_C(D^2)$ is a Morita equivalence between $A_C(S^1)$ and the trivial 2-category.

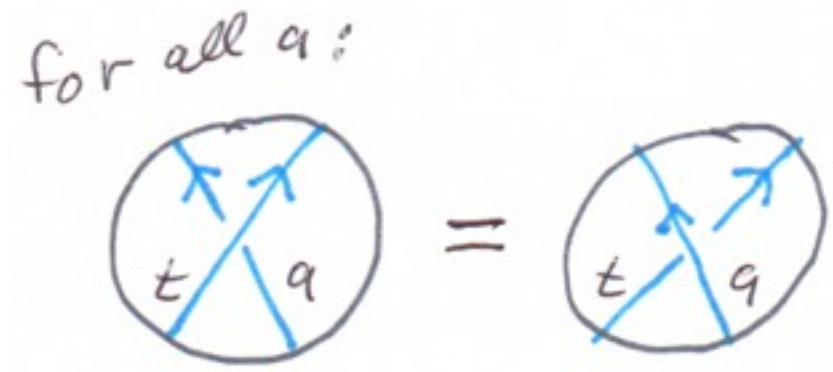
So while C is not Morita trivial, it comes close to being so.

Let E denote the 2-category $A_C(S^1)$.

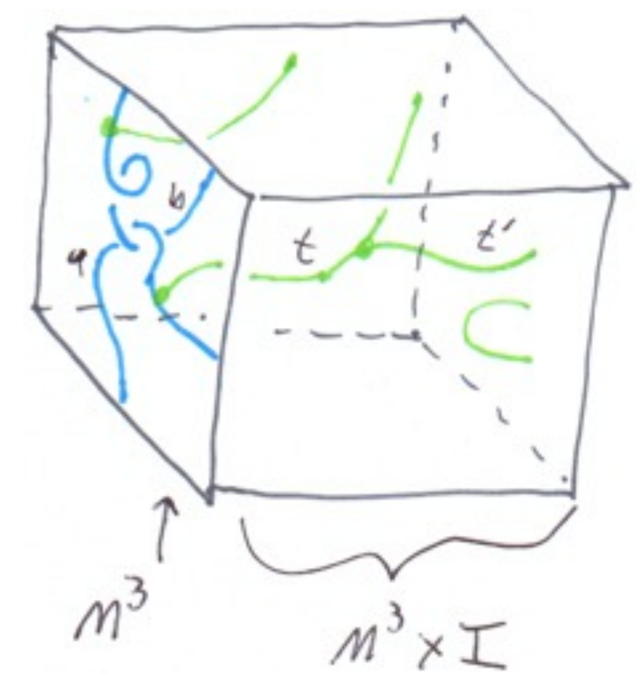
- For any closed 3-manifold M, $\dim(A_C(M)) = Z_C(M \times S^1) = Z_E(M) = 1$.
- For any closed 2-manifold Y, the number of minimal idempotents in $A_C(Y)$ is equal to $\dim(Y \times S^1) = \dim(A_E(Y)) = 1$.

Back to the general premodular case

- Let $T \subset C$ be the "transparent" subcategory of C, generated by simple objects which braid trivially with all objects of C.
- T is symmetric monoidal.
- T is trivial if and only if C is modular.

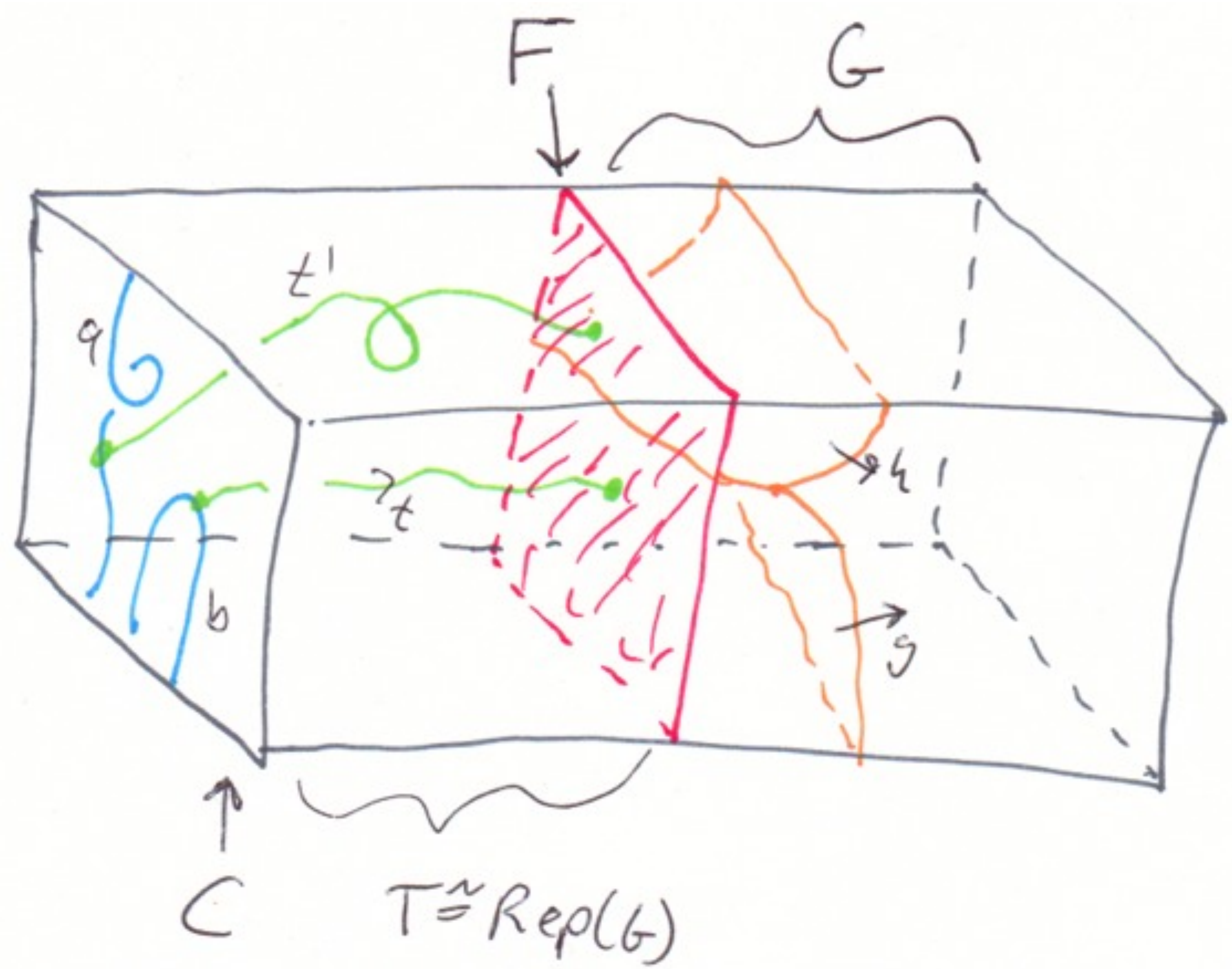


Conjecture. The 2-categories $A_C(S^1)$ and $A_T(S^1)$ are Morita equivalent. (Stronger: equivalent as modules for $A_T(S^1)$ thought of as a 3-category.)



- Because T is symmetric monoidal, we can think of it as an m-category for any m, and in particular for $m = 4$. Let T_4 denote this 4-category version of T.
- We can view the 3-category C as a module for the 4-category T_4.
- Similarly, for any $(3-k)$-manifold Y, the k-category $A_C(Y)$ is a module for the $(k+1)$-category $A_{T_4}(Y)$.

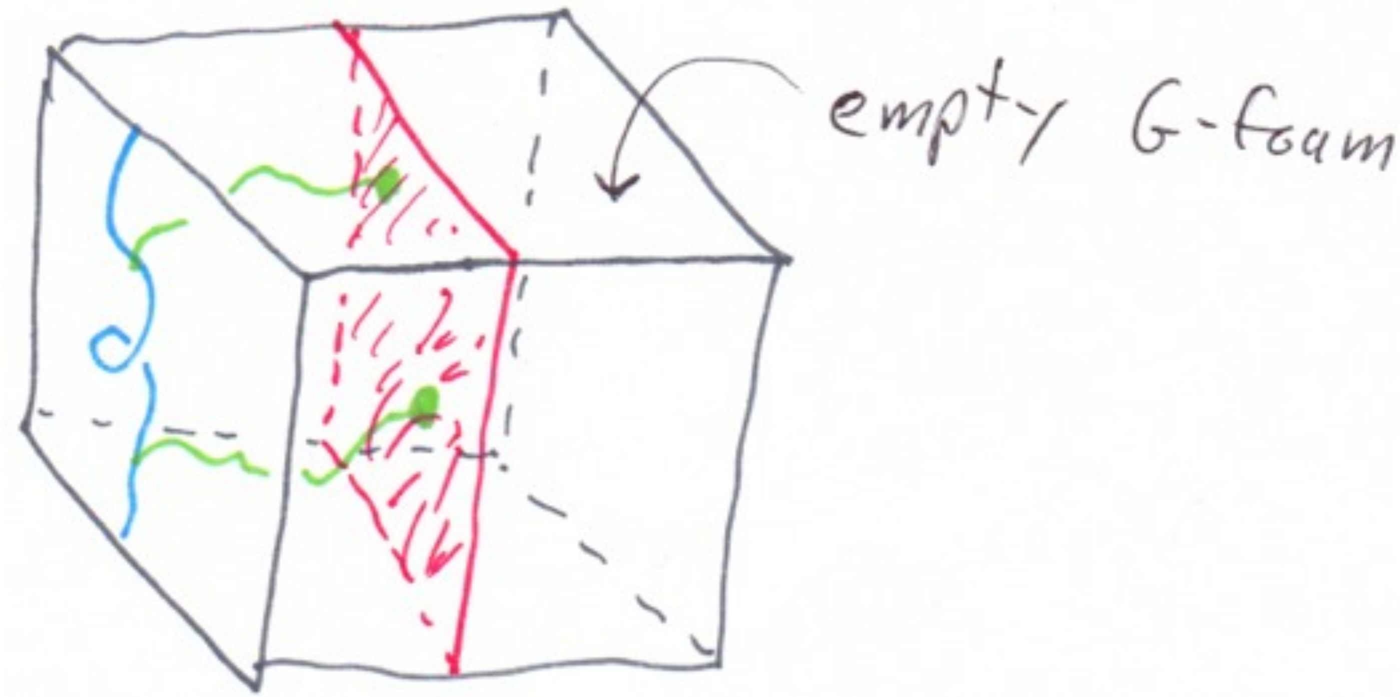
- By a theorem of Deligne/Doplicher-Roberts, as a braided category T is isomorphic to either $\text{Rep}(G)$ or $\text{Rep}(G^s)$, where G is a finite group and G^s is a finite super group.
- Taking the pivotal structure of T into account, the above classification splits into more possibilities. Roughly, there is a second $\mathbb{Z}/2$-grading on the objects corresponding to ribbon twists being ± 1.
- Let us first consider the case where $T \cong \text{Rep}(G) = R$ as a pivotal symmetric monoidal category.
- We can tensor with the Fourier R_4–G_4 bimodule F to turn the R_4 action on C into a G_4 action on $C \otimes_R F$.



- Note that having an action of G_m on an $(m-1)$-category is a very general way of saying that the finite group G acts on C. It is equivalent to having a flat connection on a bundle of $(m-1)$-categories over the classifying space BG.

- Such a connection assigns an $(m-1)$-category to each point of BG, a functor to each 1-cell of BG, a 1st order natural transformation to each 2-cell of BG, a 2nd order natural transformation to each 3-cell of BG, and so on up to m-cells. The flatness condition concerns the $(m+1)$-cells.

- If we ignore the G_4 action and just think of $C \otimes_R F$ as a 3-category, it is easy to see that it has no transparent objects (other than the trivial object) and therefore is modular. (Cf. Müger's modularization construction, for example.)



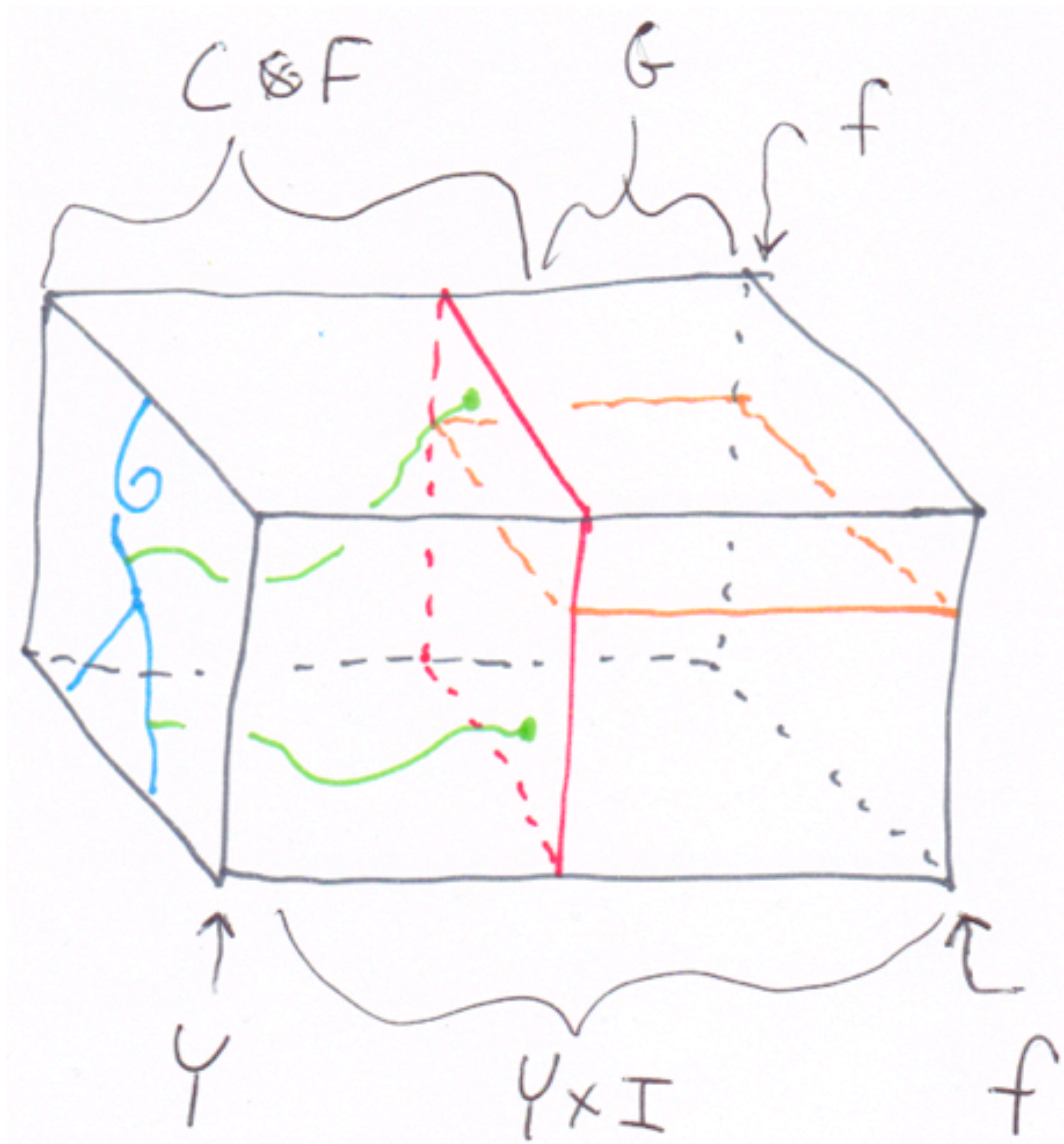
- Tensoring with the Fourier bimodule F (over the 4-categories G_4 and R_4) allows us to go back and forth between

premodular categories with transparent subcategory R	$\longleftrightarrow$	modular categories with a G_4 action

- This is very closely related to the equivariantization/deequivariantization construction of Müger and Bruguières. For example, there is a natural isomorphism

$$\mathrm{Rep}(C \otimes_R F) \cong \mathrm{Equivariantization}(\mathrm{Rep}(C)).$$

- Let Y be a $(3-k)$-manifold equipped with a map to $f : Y \to BG$. Consider $Y \times I$, with one boundary component labeled by the module $C \otimes_R F$ and the other boundary component labeled by the boundary condition f. (The interior is labeled by the 4-category G_4.) The above constructions assign a k-category to this decorated manifold. This assignment is an example of a (fully extended) Homotopy TQFT in the sense of Turaev.



- Much of the above extends to the case where the transparent subcategory of C is isomorphic to $\mathrm{Rep}(G)$ but with non-trivial braiding and/or non-trivial ribbon twist. We can still define a version of the bimodule F in these cases.

- In the case where the ribbon twist is non-trivial, we must equip the codimension-1 manifold labeled by F with a Spin structure in order to keep track of the twists.

- We can still define the isomorphisms required by the Morita equivalence, but they are not canonical – they require additional choices.