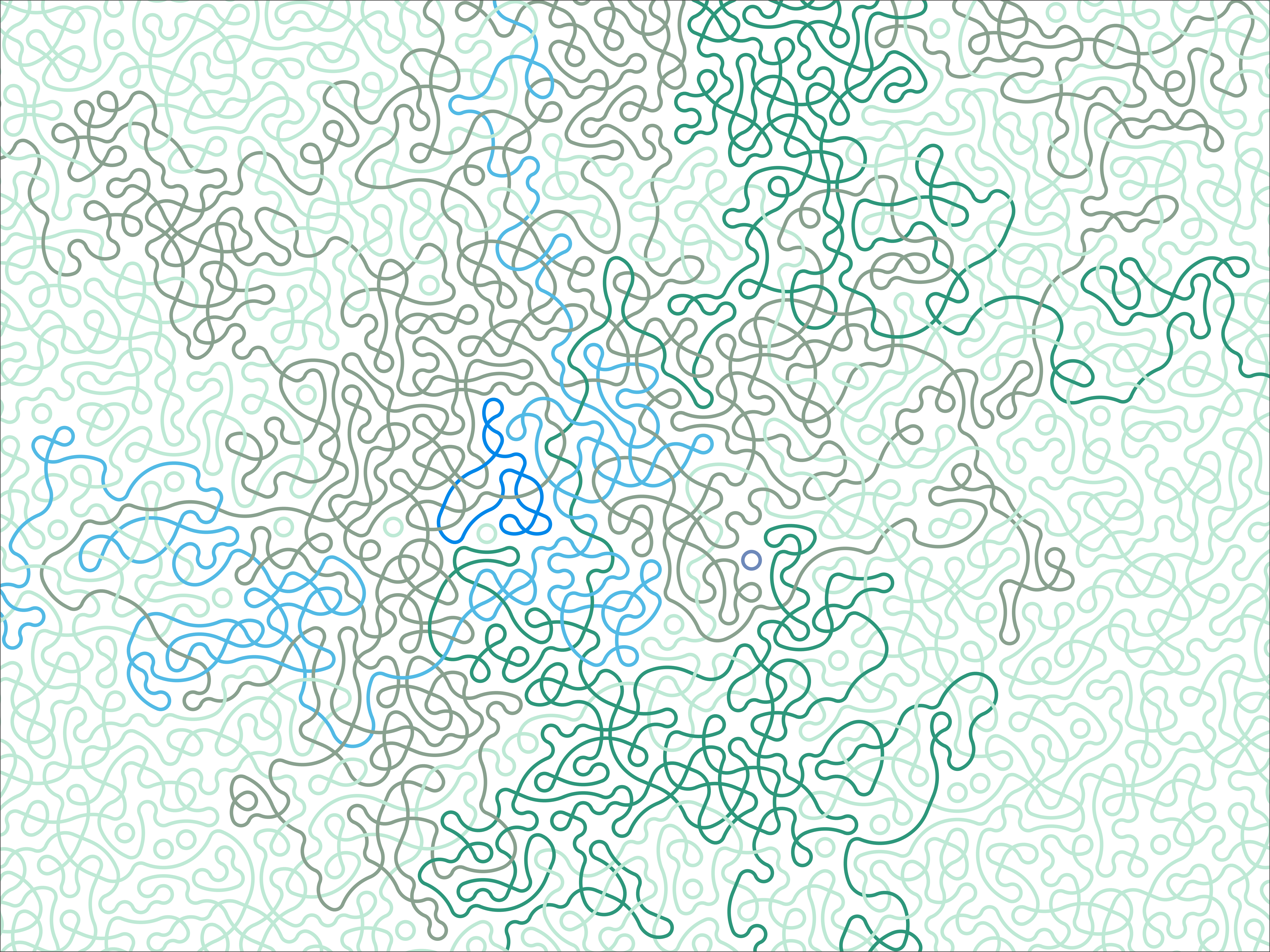


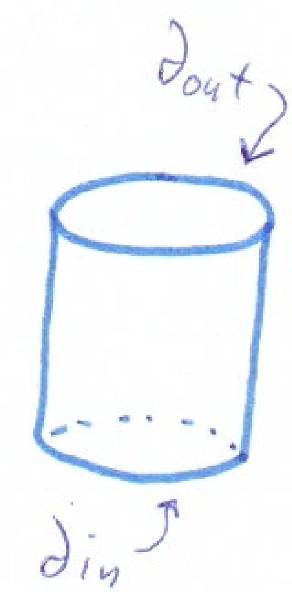
Key observation:

path integral $Z(B^{n+1}) \leftrightarrow$ local projections $\leftrightarrow$ local relations (skein-type relations)

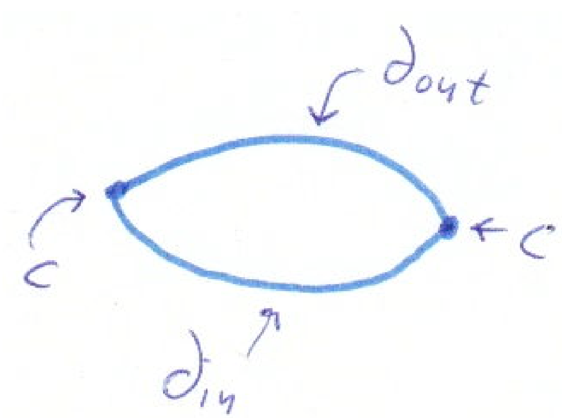
- $\mathcal{C}(X)$ = fields on X
- Path integral:

$$Z(M^{n+1}; c) \stackrel{\text{def}}{=} \int_{x \in \mathcal{C}(M;c)} T(x)$$

  Topologically invariant. $c \in \mathcal{C}(\partial M)$ is a boundary condition.
- $Z(M) : \mathcal{C}(\partial M) \to \mathbb{C}$
- $Z(M) \in \mathcal{F}(\partial M)$, where $\mathcal{F}(\partial M)$ is some appropriate space of functions from $\mathcal{C}(\partial M)$ to $\mathbb{C}$
- $Z(M) : \mathcal{F}(\partial_{\text{in}} M) \to \mathcal{F}(\partial_{\text{out}} M)$

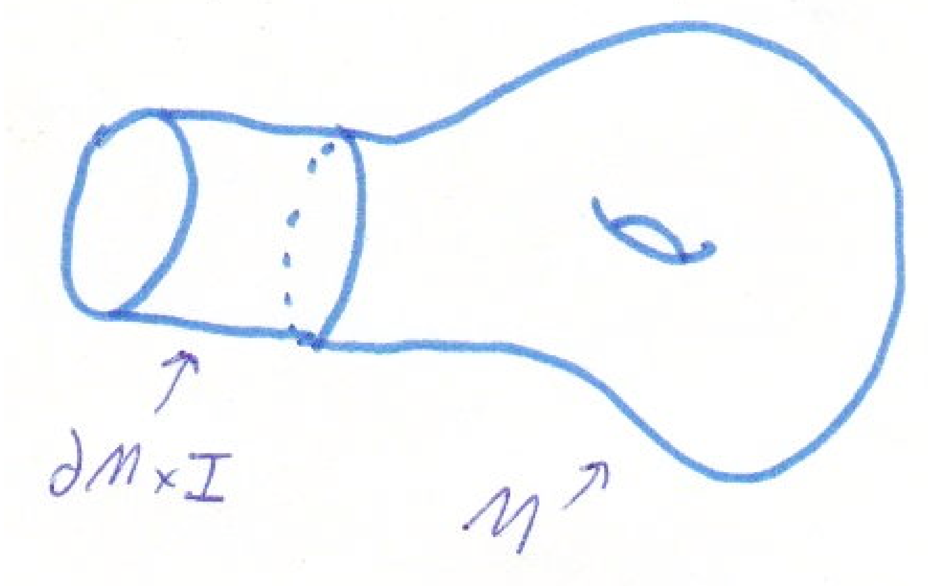


- with corners: $Z(M)_c : \mathcal{F}(\partial_{\text{in}} M; c) \to \mathcal{F}(\partial_{\text{out}} M; c)$ for all $c \in \mathcal{C}(\partial\partial_{\text{in}}) = \mathcal{C}(\partial\partial_{\text{out}})$



- in particular, $\pi_{(Y;c)} \stackrel{\text{def}}{=} Z(Y^n \times I)_c : \mathcal{F}(Y;c) \to \mathcal{F}(Y;c)$
- π_Y is a projection: $(Y \times I) \cup (Y \times I) \cong Y \times I$ implies

$$\pi_Y \circ \pi_Y = \pi_Y$$



- $M \cup (\partial M \times I) \cong M$ implies

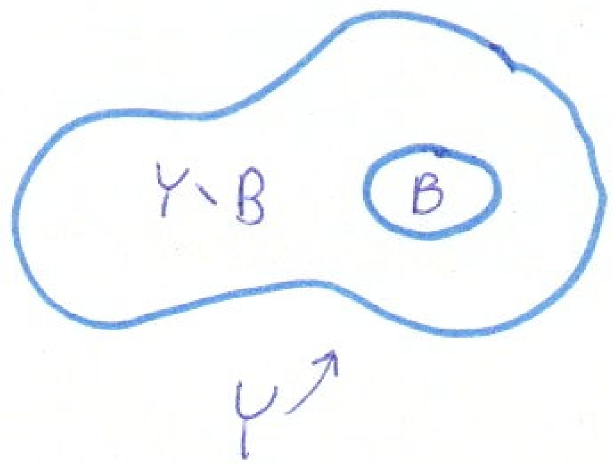
$$\pi_{\partial M}(Z(M)) = Z(M)$$

or

$$Z(M) \in \operatorname{im}(\pi_{\partial M}) \subset \mathcal{F}(\partial M)$$

- so the Hilbert space for the theory is $Z(Y) \stackrel{\text{def}}{=} \operatorname{im}(\pi_Y) \subset \mathcal{F}(Y)$

$Z(Y)$ has a local structure:



- for any $B \subset Y$, $B \cong B^n$, we have

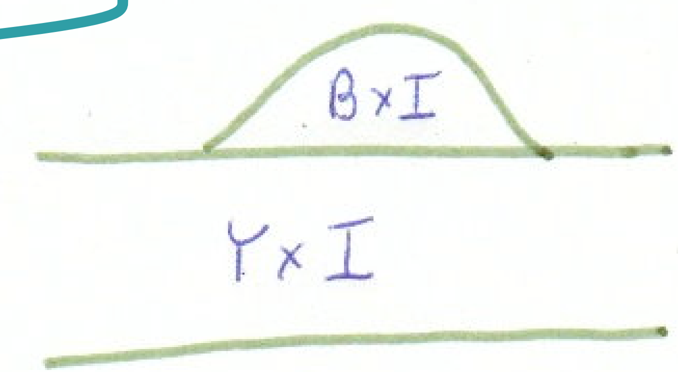
$$\mathcal{F}(Y) \cong \bigoplus_{c \in \mathcal{C}(\partial B)} \mathcal{F}(B; c) \otimes \mathcal{F}(Y \setminus B; c)$$

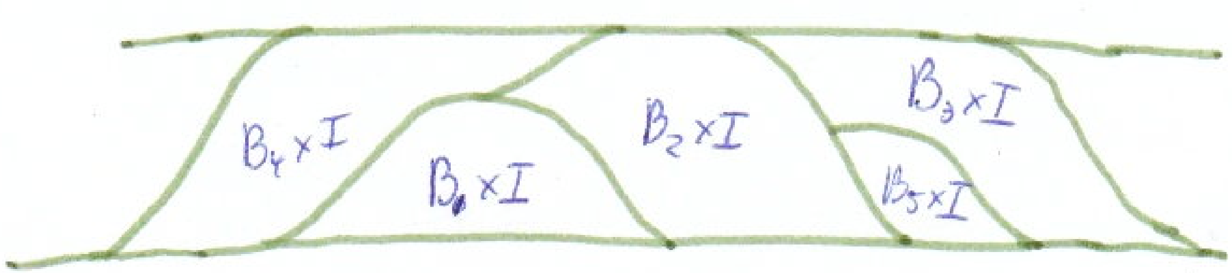
- define $\pi_B : \mathcal{F}(Y) \to \mathcal{F}(Y)$ by

$$\pi_B \stackrel{\text{def}}{=} \bigoplus_{c \in \mathcal{C}(\partial B)} \pi_{(B;c)} \otimes \text{id}$$

- $(Y \times I) \cup (B \times I) \cong Y \times I$ implies $\pi_B \circ \pi_Y = \pi_Y$ implies $Z(Y) \subset \text{im}(\pi_B)$, for all $B \subset Y$
- let $\{B_i\}$ be an open cover of Y; then $Y \times I \cong B_0 \times I \cup \cdots \cup B_k \times I$ implies $\pi_Y = \pi_{B_1} \circ \cdots \circ \pi_{B_k}$ implies $Z(Y) = \text{im}(\pi_{B_0}) \cap \cdots \cap \text{im}(\pi_{B_k})$
- it follows that

$$\boxed{Z(Y) = \bigcap_{B \subset Y} \text{im}(\pi_B)}$$





It is usually more convenient to work with dual spaces.

- $A(Y) \stackrel{\text{def}}{=} Z(Y)^* = \mathbb{C}[\mathcal{C}(Y)]/U(Y)$, where

$$U(Y) \stackrel{\text{def}}{=} \{a \in \mathbb{C}[\mathcal{C}(Y)] \mid f(a) = 0 \text{ for all } f \in Z(Y)\}$$

- define $LR(Y)$, the space of local relations on $\mathbb{C}[\mathcal{C}(Y)]$, to be the span of all $u \bullet r$, where $B \subset Y$, $B \cong B^n$, $u \in U(B)$, $r \in \mathcal{C}(Y \setminus B)$, and $u \bullet r$ denotes the gluing of u and r

- then the locality result for $Z(Y)$ implies that

$$\boxed{A(Y) \cong \mathbb{C}[\mathcal{C}(Y)]/LR(Y);}$$

  in other words, the dual of the Hilbert space is a generalized skein module

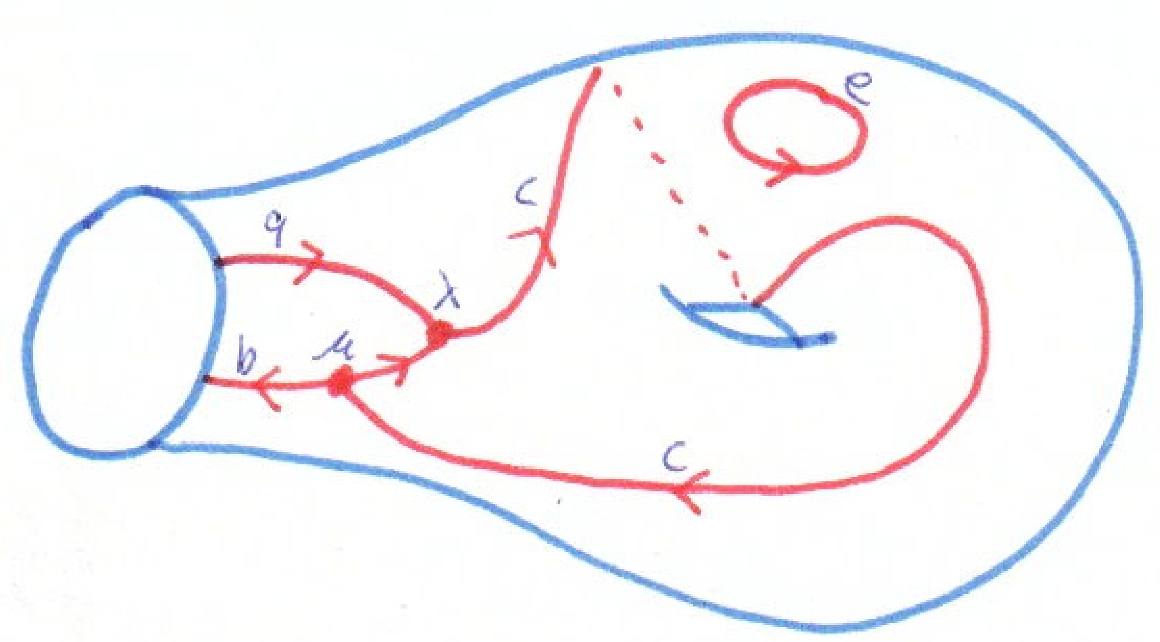
Now for the rigorous definitions:

A system of fields for manifolds of dimension $\leq n$ consists of:

- a collection of functors $\mathcal{C}_k : \mathcal{M}_k \to \mathbf{Set}$, $k \leq n$, where $\mathcal{M}_k$ denotes the category of k-manifolds and homeomorphisms (PL, say)
- additional data [...] (see below)
- satisfying conditions [...] (see below)

Main examples:

- $\mathcal{C}(X) = \{\text{maps } X \to C\}$, e.g. $C = B\Gamma$, where Γ is a finite group
- $\mathcal{C}(X) = \{\text{decorated cell complexes } \subset X\}$



The rest of the "fields" definition:

- restriction maps $\mathcal{C}(X) \to \mathcal{C}(\partial X)$ (natural transformation of functors)
- orientation reversal maps $\mathcal{C}(X) \to \mathcal{C}(-X)$ (natural transformation of functors)
- compatibility with monoidal structure $\mathcal{C}(X \sqcup W) \cong \mathcal{C}(X) \times \mathcal{C}(W)$
- gluing along $Y \subset \partial X$, $-Y \subset \partial W$ corresponds to a fibered product

$$\begin{tikzcd} & \mathcal{C}(X) \arrow[dr] & \\ \mathcal{C}(X \cup_Y W) \arrow[ur] \arrow[dr] & & \mathcal{C}(Y) \\ & \mathcal{C}(W) \arrow[ur] & \end{tikzcd}$$

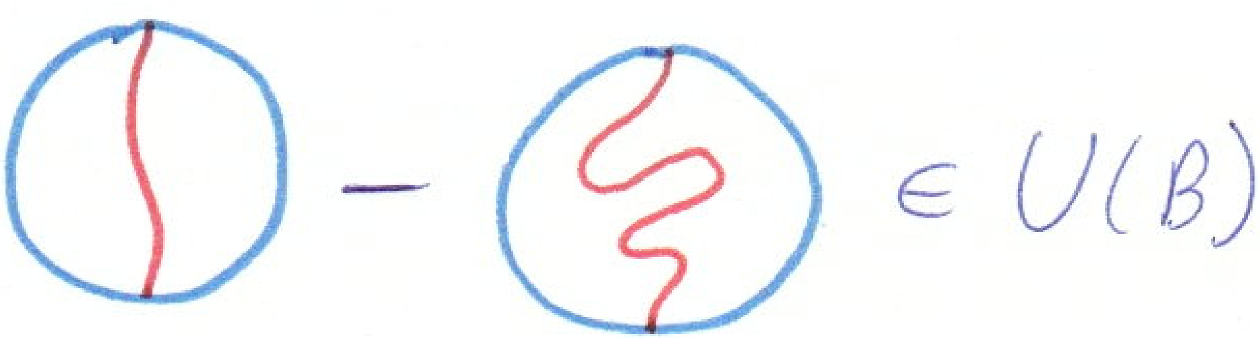
  (and similarly for self-gluing, gluing with corners) (up to isotopy supported near Y)

- "product with I" maps $\mathcal{C}(Y) \to \mathcal{C}(Y \times I)$; fiber-preserving homeos of $Y \times I$ act trivially on image

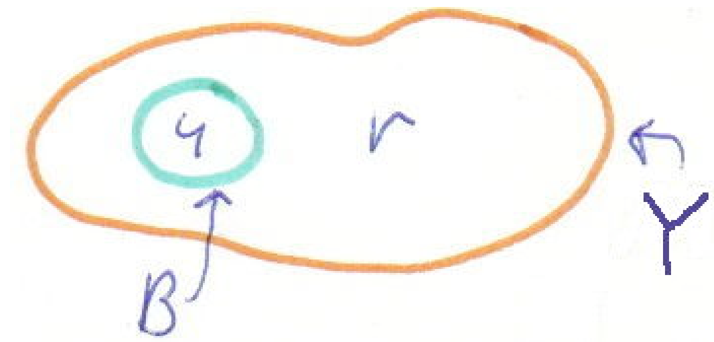
Definition of "local relations":

For each n-manifold $B \cong B^n$ and $c \in \mathcal{C}(\partial B)$, a subspace $U(B;c) \subset \mathbb{C}[\mathcal{C}(B;c)]$, preserved under homeomorphisms, such that

- local relations are at least as strong as isotopy: for all $a, b \in \mathcal{C}(B)$ with a isotopic to b (pseudo-isotopic or extended isotopic), we have $a - b \in U(B)$.
- local relations are an ideal with respect to gluing: for all $B = B_1 \cup B_2$, $u \in U(B_1)$, $r \in \mathcal{C}(B_2)$, we have $u \bullet r \in U(B)$

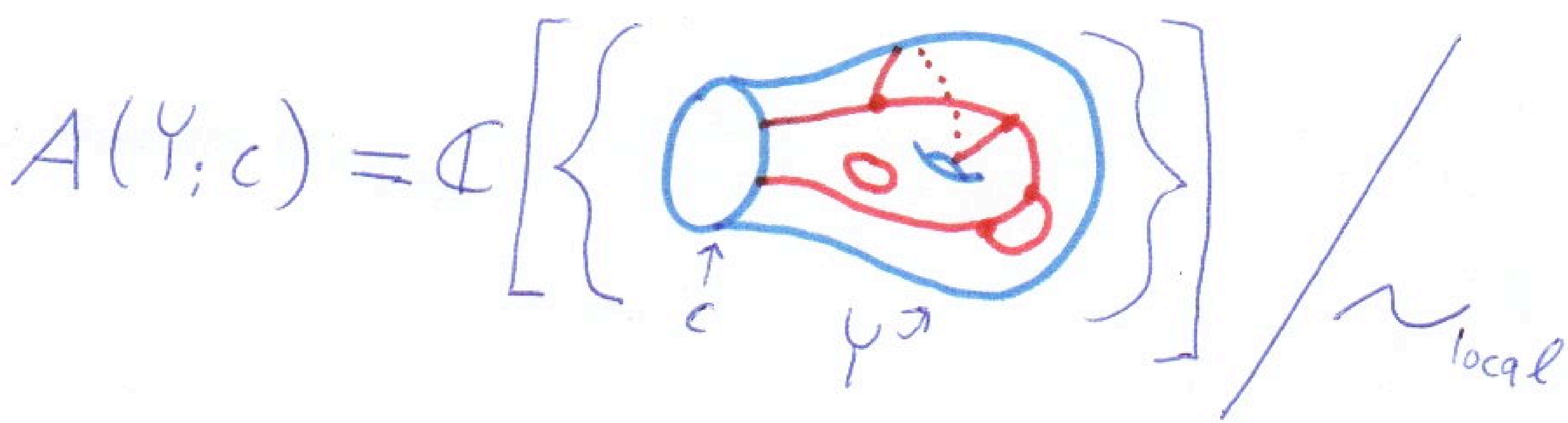


Basic constructions, dimension n



$$A(Y^n; c) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}(Y)]/LR(Y)$$

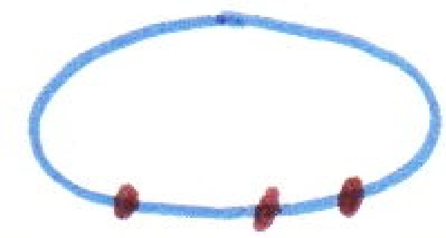
where $c \in \mathcal{C}(\partial Y)$ is a boundary condition and $LR(Y)$ is the span of all $u \bullet r$, $B \subset Y$, $u \in U(B)$, $r \in \mathcal{C}(Y \setminus B)$

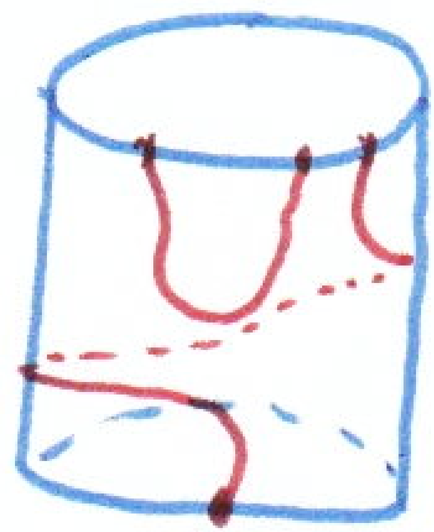


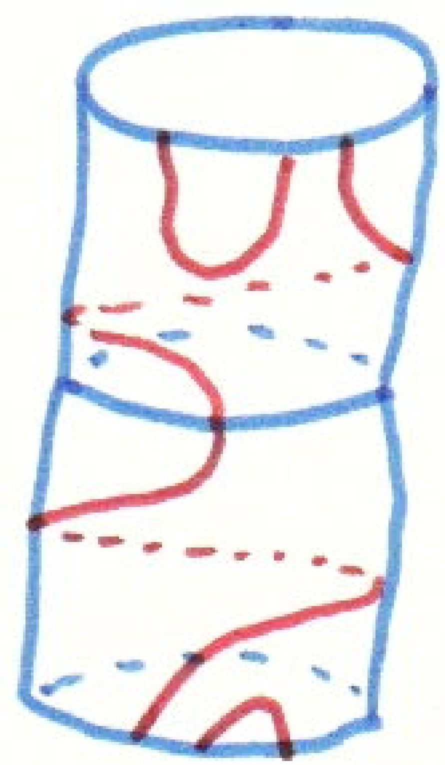
Basic constructions, dimension n-1

$A(W^{n-1}; c)$ is a *-1-category:

- objects: $\mathcal{C}(W; c)$
- morphisms from a to b: $A(W \times I; a, b)$
- composition: gluing







Basic constructions, dimension n-2

$A(Q^{n-2}; c)$ is a pivotal 2-category:

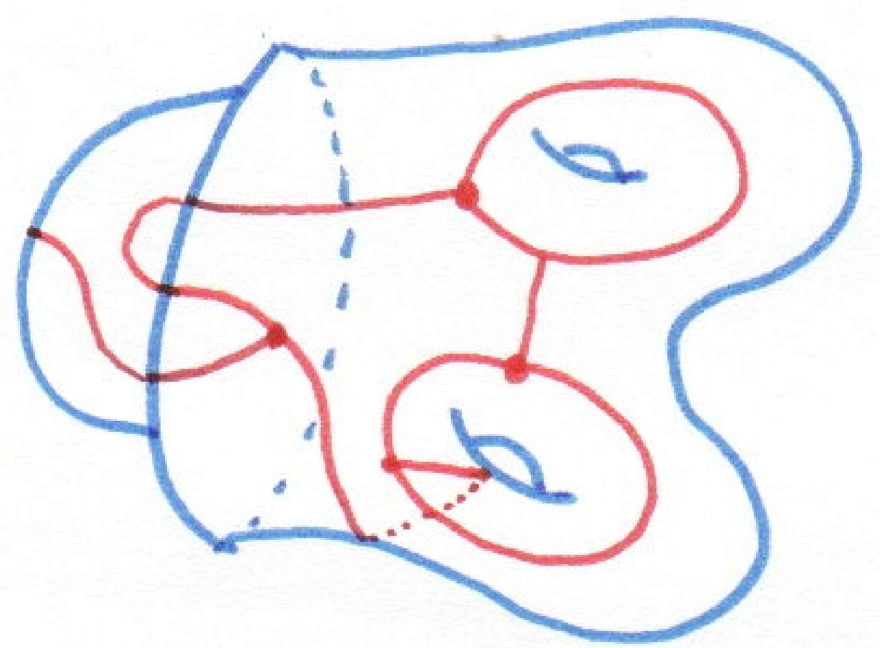
- 0-morphisms: $\mathcal{C}(Q; c)$
- 1-morphisms from a to b: $\mathcal{C}(Q \times I; a, b)$
- 2-morphisms from e to f: $A((Q \times I) \times I; e, f)$
- composition: gluing

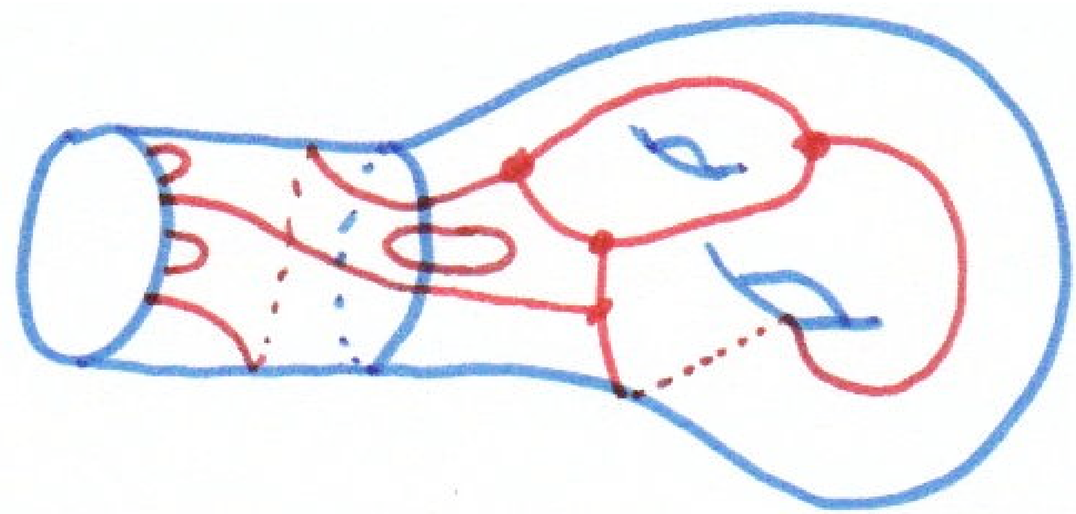
And so on: $A(X^{n-k})$ is a linear k-category with strong duality. For $j < k$, the j-morphisms are $\mathcal{C}(X \times I^j; \cdot)$. The k-morphisms are $A(X \times I^k; \cdot)$.

Manifolds afford representations of their boundary categories

$\{A(W^{n-k}; c)\}$, where c runs through all of $\mathcal{C}(\partial W)$, affords a representation of the $k+1$-category $A(\partial W)$, via gluing of collars.

More generally, let $Z \subset \partial W$ be a codimension-0 submanifold, and $b \in \mathcal{C}(\partial W \setminus Z)$. Then $\{A(W^{n-k}; b, c)\}$, where c runs through all of $\mathcal{C}(Z, \partial b)$, affords a representation of the $k+1$-category $A(Z; \partial b)$, via gluing of partial collars.





Gluing

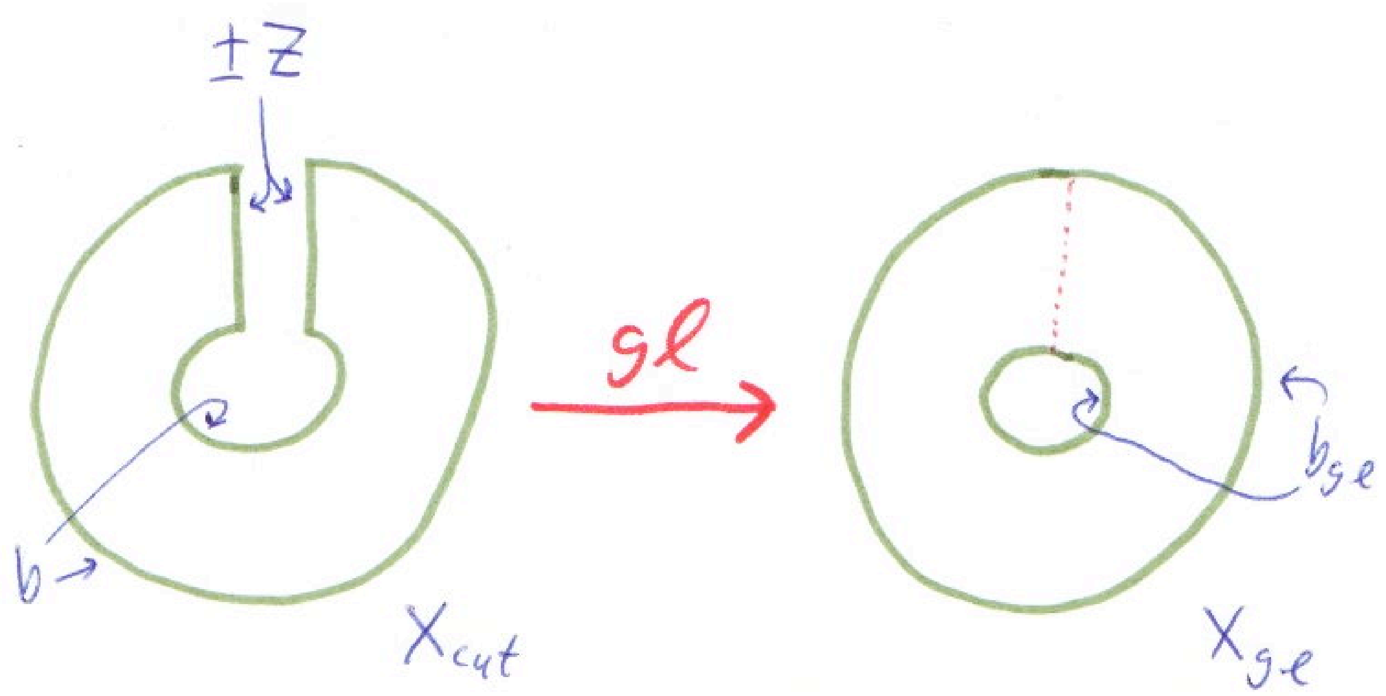
Let X_{cut} be an $n-k$-manifold, with $Z \sqcup -Z$ embedded as a codim-0 submanifold of ∂X_{cut}. Identifying the copies of Z yields a manifold X_{gl}.

Theorem:

$$A(X_{\text{gl}}; b_{\text{gl}}) \cong \bigotimes_{A(Z;\partial b)} \{A(X_{\text{cut}}; b, \cdot)\}$$

(k times categorified coend)

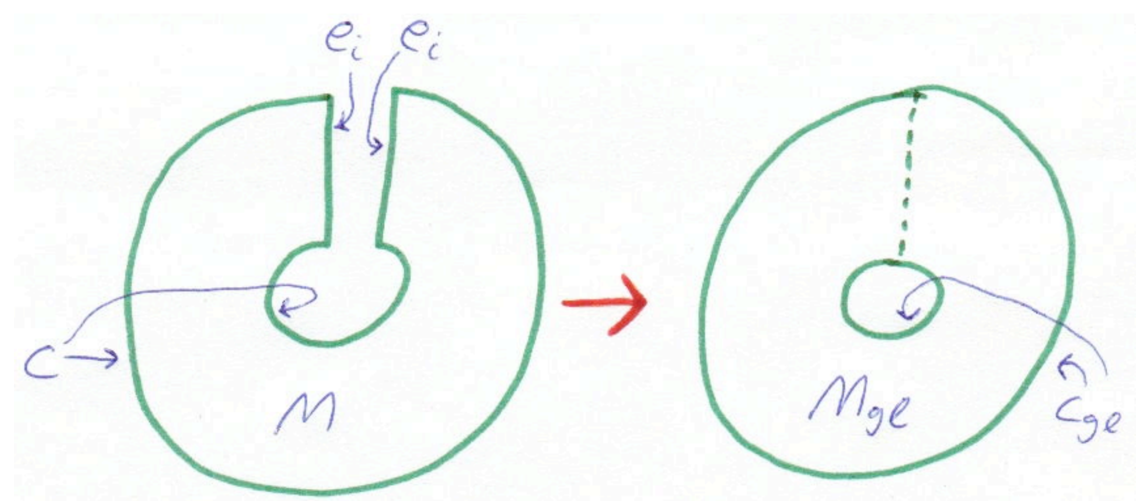
(Drinfeld double is a special case of the once categorified coend. Drinfeld center is a special case of the once categorified end.)



What about Z?

- $Z(Y^n) \stackrel{\text{def}}{=} A(Y)^* = \{f : \mathbb{C}[\mathcal{C}(Y)] \to \mathbb{C} \mid f(u \bullet r) = 0 \text{ for all } B, u, r \text{ as above}\}$
- $Z(W^{n-1}) \stackrel{\text{def}}{=} \mathrm{Rep}(A(W))$ (i.e. functors from $A(W)$ to **Vect**)
- and in general, $Z(X^{n-k}) \stackrel{\text{def}}{=} \mathrm{Rep}(A(X))$
- for $\dim X \leq n$, we have $Z(X) \in Z(\partial X)$

What about dimension n+1?



What we want from a path integral:

- For all M^{n+1}, $Z(M) \in Z(\partial M)$, i.e.

$$Z(M) : A(\partial M) \to \mathbb{C}$$

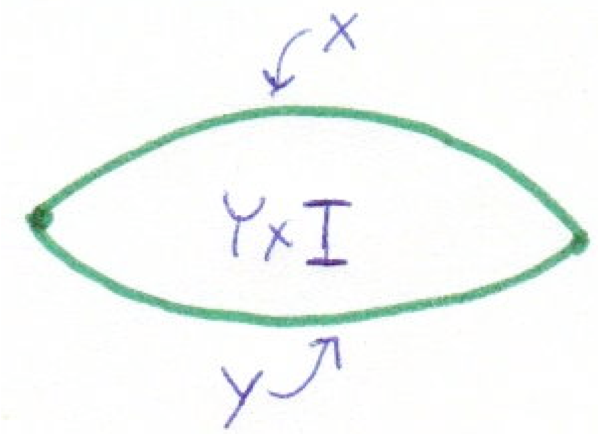
- satisfying the gluing formula

$$Z(M_{\mathrm{gl}})(c_{\mathrm{gl}}) = \sum_i Z(M)(e_i \bullet e_i \bullet c) \frac{1}{\langle e_i, e_i \rangle},$$

  where e_i runs through an *orthogonal* basis of $A(Y; \partial c)$

- and where the (non-degenerate) inner products of $A(Y^n; b)$ are related to the path integral via

$$\langle x, y \rangle = Z(Y \times I)(x \bullet y)$$



Theorem. Suppose

1. there exists $z \in Z(S^n)$ such that the induced inner product $A(B^n; c) \otimes A(B^n; c) \to \mathbb{C}$ given by $a \otimes b \mapsto z(a \bullet b)$ is positive definite for all $c \in \mathcal{C}(S^{n-1})$; and

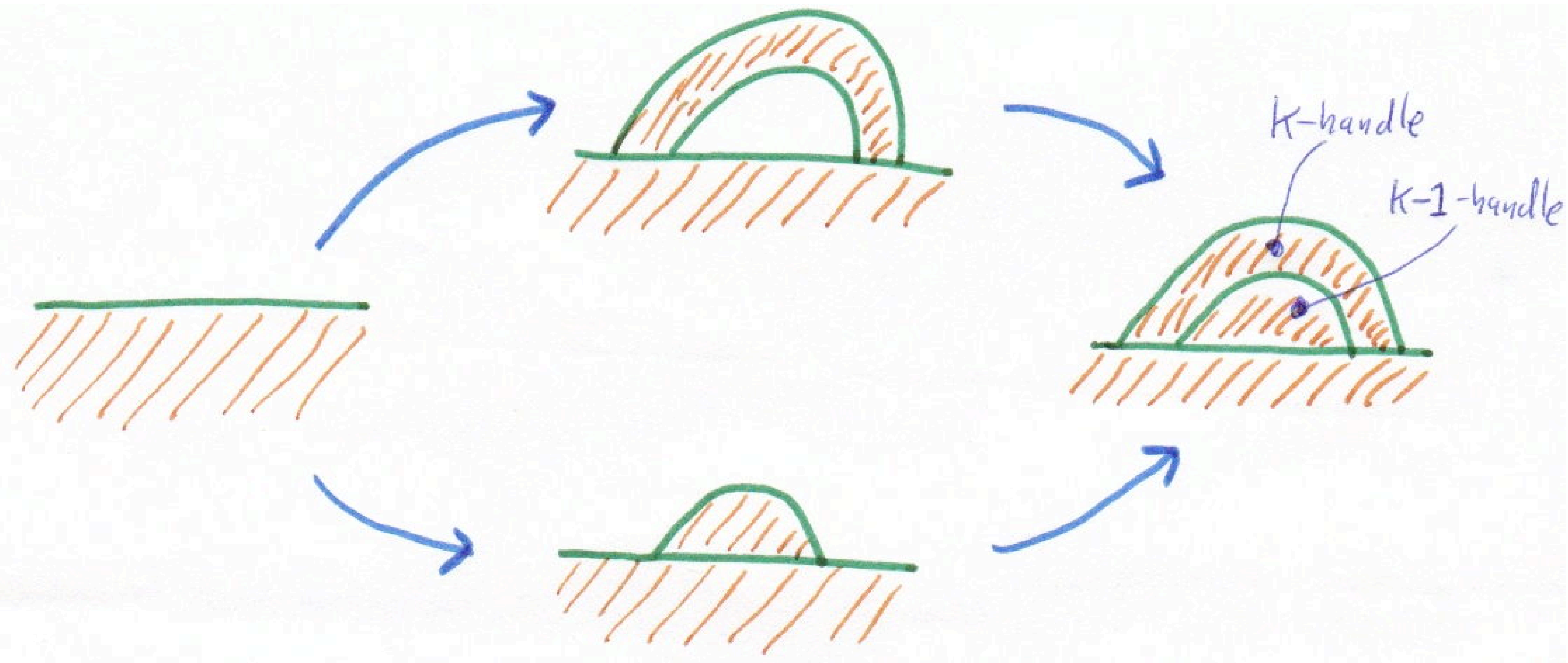
2. $\dim A(Y^n; c) < \infty$ for all n-manifolds Y and all $c \in \mathcal{C}(\partial Y)$.

Then there exists a unique path integral $Z(M^{n+1}) \in Z(\partial M)$ (for all $n+1$-manifolds M) satisfying the the above conditions and such that $Z(B^{n+1}) = z$.

Sketch of proof:

- Choose a handle decomposition of M. Adding the handles one at a time (lowest index first) determines $Z(M)$ via the gluing formula. This proves uniqueness.

- To prove existence, must show that the computation of the previous step does not change if we cancel a pair of handles. This follows from the more general fact (lemma) that the gluing formula is associative. So we can add the canceling pair of handles in reverse order, but this is equivalent to adding partial collars, and hence has no effect on the computation.

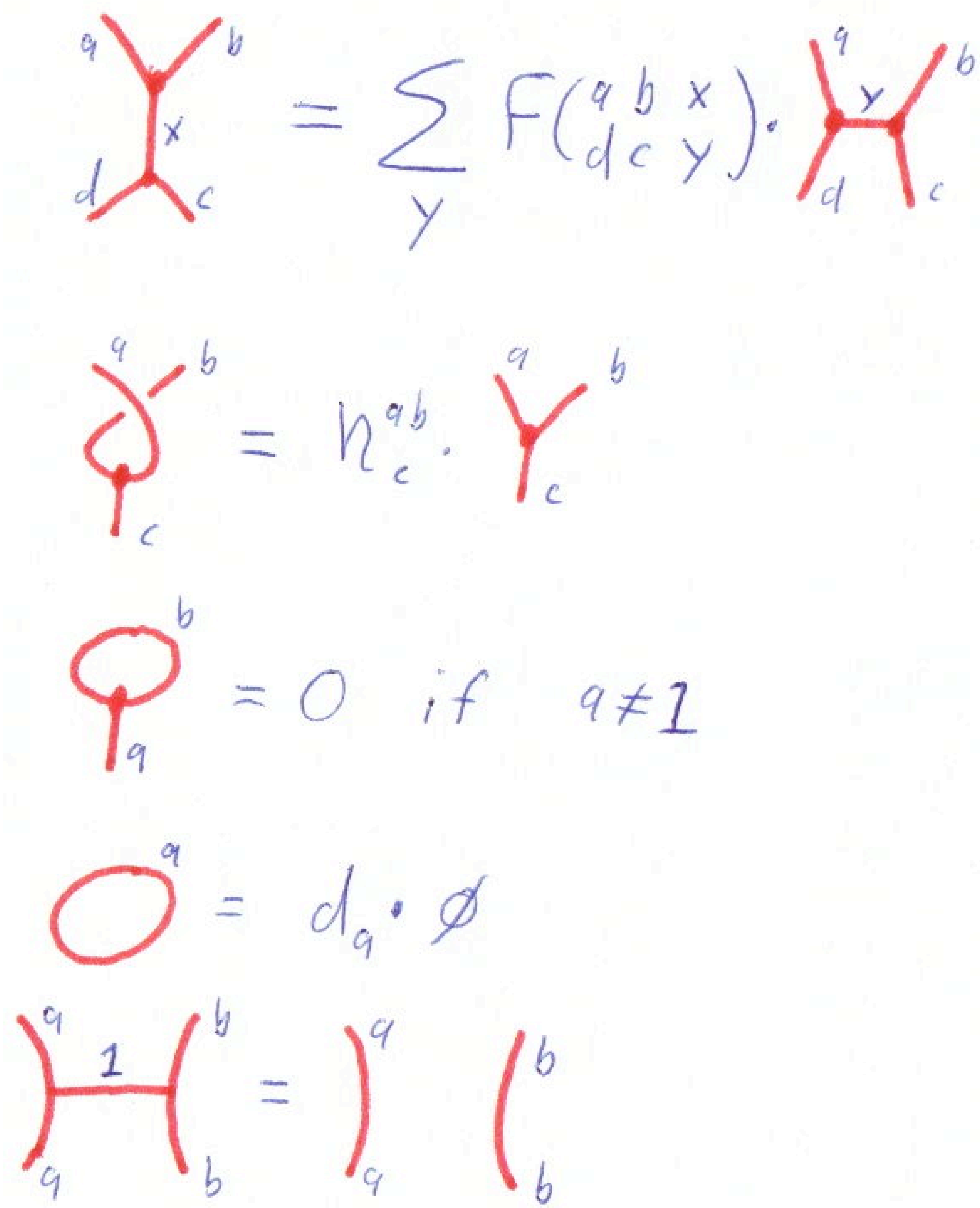


Examples...

Fields	Local Relation	State Sum
Maps into BG (G a finite group) n = arbitrary	Homotopy of maps	Dijkgraaf-Witten sum on a triangulation
Pictures based on a pivotal 2-category n = 2	Isotopy plus relations coming from the category	Turaev-Viro sum
Pictures based on a ribbon category (a disklike 3-category) n=3	Isotopy plus relations coming from the category	For a generic cell handle decomposition of a 4-manifold, the Crane-Yetter state sum
[same as above]	[same as above]	For 2-handles attached to the 4-ball, the Witten-Reshetikhin-Turaev surgery formula
[same as above]	[same as above]	For a “special spine” of a 4-manifold, the Turaev shadow state sum

Fields and relations from a modular ribbon category

- $\mathcal{C}(M^3) = \{$3-valent labeled ribbon graphs in $M\}$
- local relations:



Inductively determine inner products for attachment regions of i-handles $S^{i-1} \times B^{4-i}$. Start the induction with $Z(B^4) : A(S^3) \to \mathbb{C}$,

$$Z(B^4)(x) = \lambda \cdot ev_{\mathrm{std}}(x)$$

IP

$$\langle \textcircled{a}, \textcircled{a} \rangle_{A(B^3)} = Z(B^4)(\bigcirc^{a}) = \lambda d_a$$

GL

$$\Rightarrow Z(S^1 \times B^3)(\ell_a \ell_b) = \begin{cases} Z(B^4)(\bigcirc^{a}) \cdot \dfrac{1}{\langle \textcircled{a}, \textcircled{a} \rangle} = 1 & \text{if } a = b \\ 0 \quad , & a \neq b \end{cases}$$

IP

$$\Rightarrow \langle \ell_a, \ell_b \rangle_{A(S^1 \times B^2)} = \delta_{ab}$$

$$\boxed{D := \sqrt{\textstyle\sum d_a^2}}$$

GL

$$\Rightarrow Z(S^2 \times B^2)(\emptyset) = \sum_a Z(\underset{\substack{\uparrow \\ \text{0-handle}}}{B^4})(\bigcirc^{a}) \cdot Z(\underset{\substack{\uparrow \\ \text{2-handle}}}{B^4})(\bigcirc^{a}) = \lambda^2 \cdot \sum_a d_a^2 = \lambda^2 D^2$$

IP

$$\Rightarrow \langle \emptyset, \emptyset \rangle_{A(S^2 \times I)} = \lambda^2 D^2$$

GL

$$\Rightarrow Z(S^3 \times I)(\emptyset) = Z(\underset{\substack{\uparrow \\ \text{0-handle}}}{B^4})(\emptyset) \cdot Z(\underset{\substack{\uparrow \\ \text{3-handle}}}{B^4})(\emptyset) \cdot \frac{1}{\lambda^2 D^2} = \frac{1}{D^2}$$

IP

$$\Rightarrow \langle \emptyset, \emptyset \rangle_{A(S^3)} = \frac{1}{D^2}$$

- If $\lambda = D^{-1}$, then $Z(S^0 \times B^4) = Z(B^1 \times S^3)$.
- If, in addition,

  then $\dim(A(S^1 \times S^2)) = 1$ and $Z(S^1 \times B^3) = Z(B^2 \times S^2)$
- So $Z(W^4)$ depends only on the bordism class of W (i.e. only on its signature)
- This implies that $\dim(A(M^3)) = 1$ for any closed M.
- The state sum corresponding to generic cell decomposition of W^4_{closed} is (more or less) the Crane-Yetter state sum.
- (Things are looking boring and disappointing, but...)

- Applying the gluing formula to $W^4 = $ 0-handle $\cup$ 2-handles, we get the Witten-Reshetikhin-Turaev surgery formula for ∂W.

- More generally, we find that

$$Z_{WRT}(X) = Z(\partial^{-1} X)$$

  for $dim(X) = 3, 2, 1$.

- The extra structure on X needed to choose X modulo 4-dimensional bordism corresponds precisely to the extra structure needed to make the older definitions of Z_{WRT} well-defined.

- $Z_{WRT}(pt) = Z(\partial^{-1} pt)$??

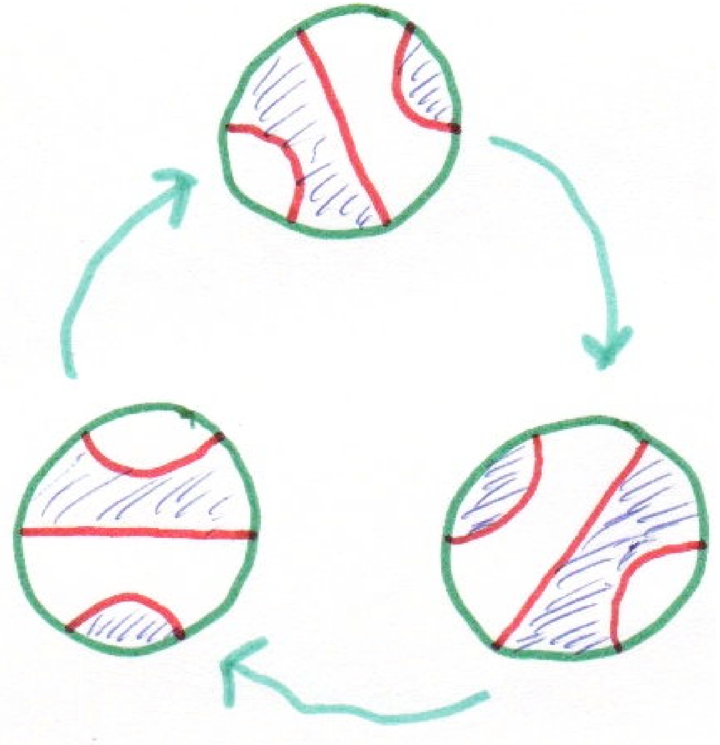
- But $Z(pt)$ is easy to define: it's the 3-category Rep(C), where C is the ribbon 3-category with which we started the construction.

Part 2

Goal: Apply the machinery from the previous hour to interesting new examples

Contact structures as a TQFT (n=3)

- $\mathcal{C}(M^3) = \{\text{contact structures on } M\}$
- $\mathcal{C}(Y^2) = \{\text{germs of contact structures on } Y \times [-\varepsilon, \varepsilon]\}$
- local relations: (1) isotopy, (2) overtwisted disk ~ 0.
- A basis for $A(M^3, C)$ is the set of tight contact structures on M restricting to c on ∂M, modulo isotopy.

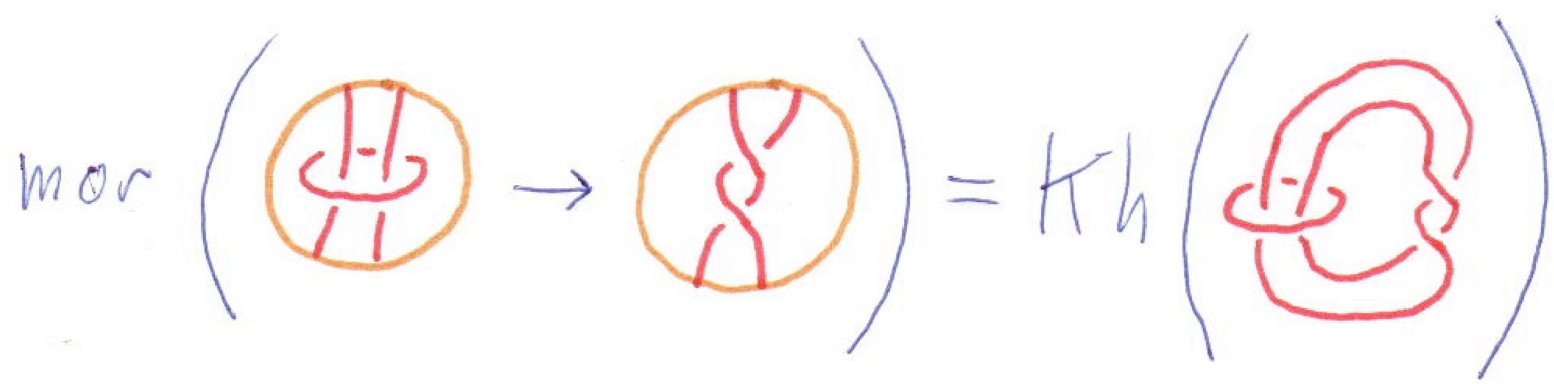


Khovanov homology as a TQFT (n=4)

Khovanov homology has the structure of a disk-like 4-category:

- 0-morphisms: nothing in B^0
- 1-morphisms: nothing in B^1
- 2-morphisms: points in B^2
- 3-morphisms: tangles in B^3
- 4-morphisms $T_1 \to T_2$: $\mathrm{Kh}(\overline{T_1} \bullet T_2)$ (a bigraded $\mathbb{C}[\alpha]$ module)

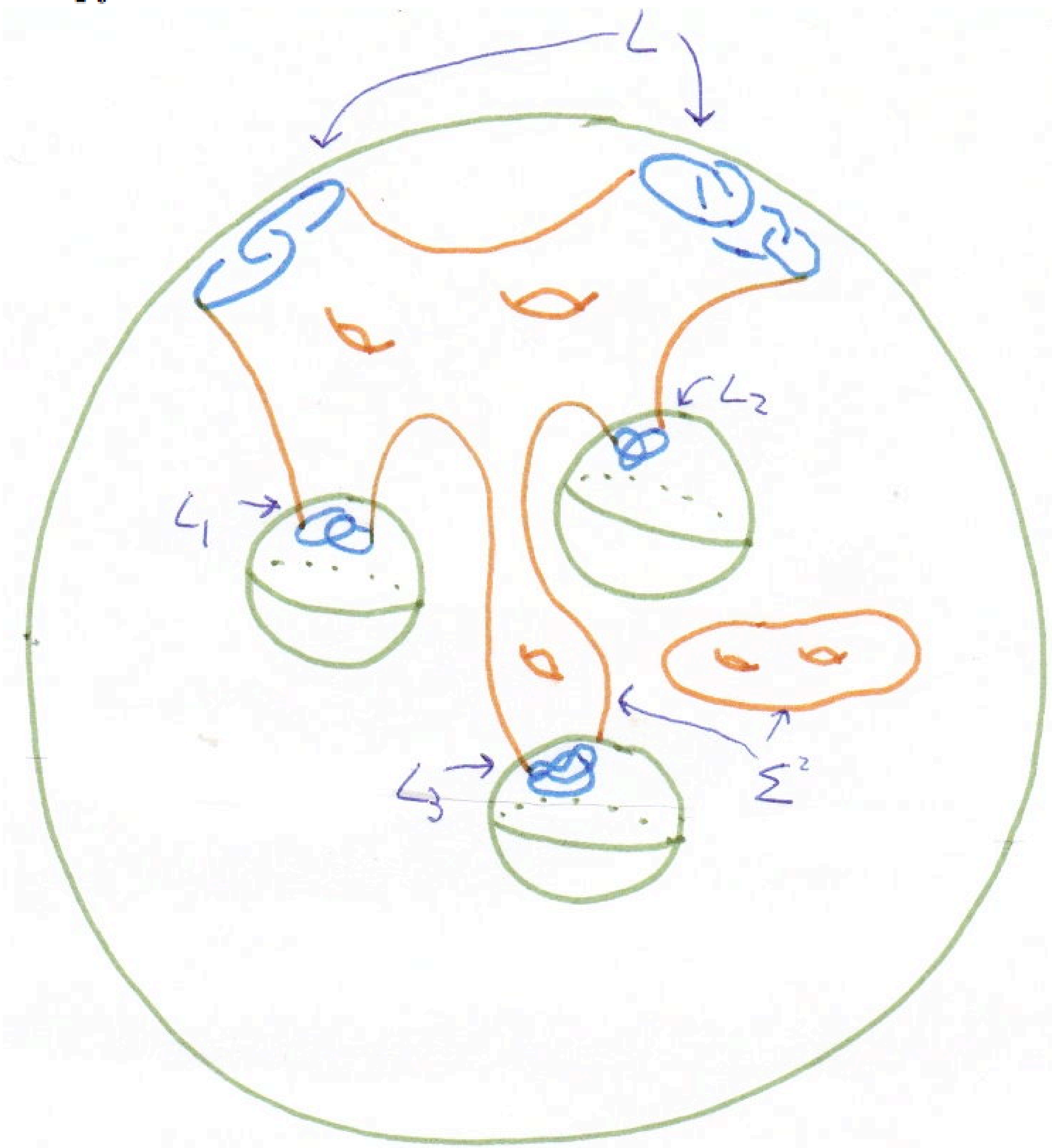
Composition of 0, 1, 2, 3-morphisms is obvious, as is duality.
Composition and duality of 4-morphisms...



Operadish product on Kh:

$$\mathrm{Kh}(L_1) \otimes \cdots \otimes \mathrm{Kh}(L_k) \to \mathrm{Kh}(L)$$

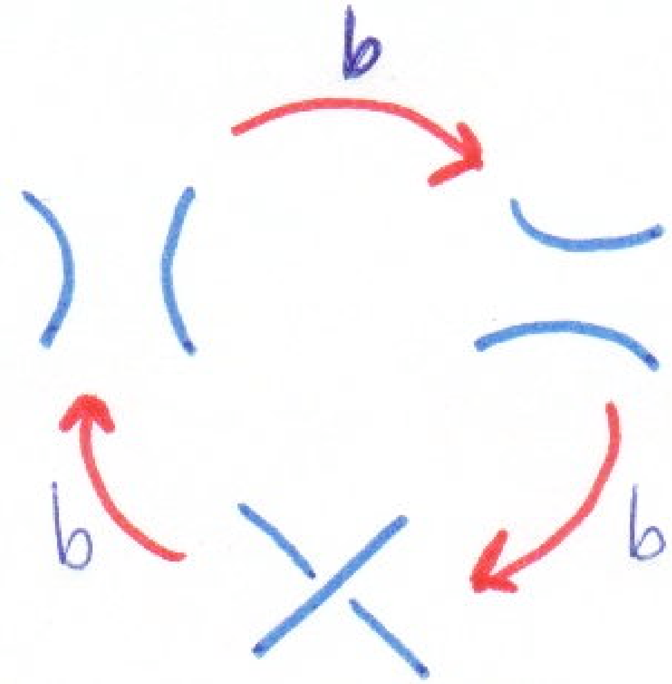
Invariant under isotopy.



Applying the above constructions, we get a 4+1-dimensional TQFT (minus the 5-dimensionsal part). It assigns a bigraded $\mathbb{C}[\alpha]$ module $A_{\mathrm{Kh}}(W^4; L)$ to each 4-manifold W. $A_{\mathrm{Kh}}(B^4; L) \cong \mathrm{Kh}(L)$.

How to calculate?

For $\mathrm{Kh}(L)$ (a.k.a. $A_{\mathrm{Kh}}(B^4; L)$), one makes extensive use of the exact triangle (long exact sequence)



The quotient in the definition of $A_{\mathrm{Kh}}(W^4; L)$ breaks the exactness. (The contact TQFT has a similar exact triangle, which also breaks.)

One solution: replace ordinary tensor products (over 1-categories) with derived tensor products. But then we would need to show that the answer did not depend on how we cut W up into 4-balls.

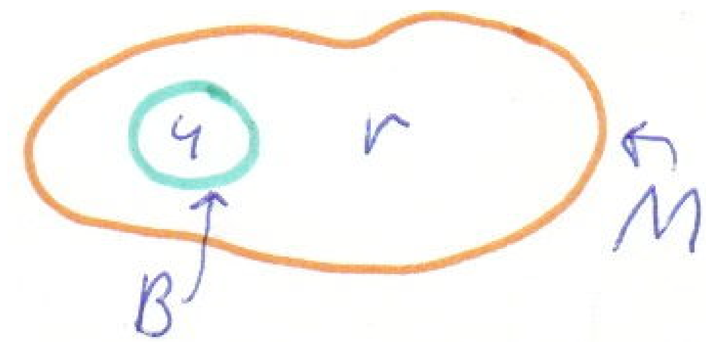
We prefer a solution which is manifestly well-defined and functorial...

Blob homology

$$\left.\begin{array}{r} n\text{-manifold } M \\ n\text{-category } C \end{array}\right\} \longrightarrow \text{chain complex } \mathcal{B}_*(M,C)$$

$$A_C(M) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}_C(M)]/\mathbb{C}[\{(B,u,r)\}]$$

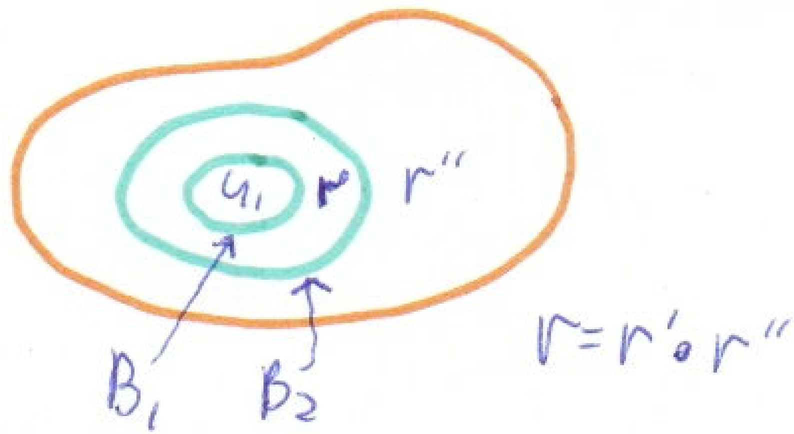
where $B \subset M$, $u \in U(B)$, and $r \in \mathcal{C}(M \setminus B)$



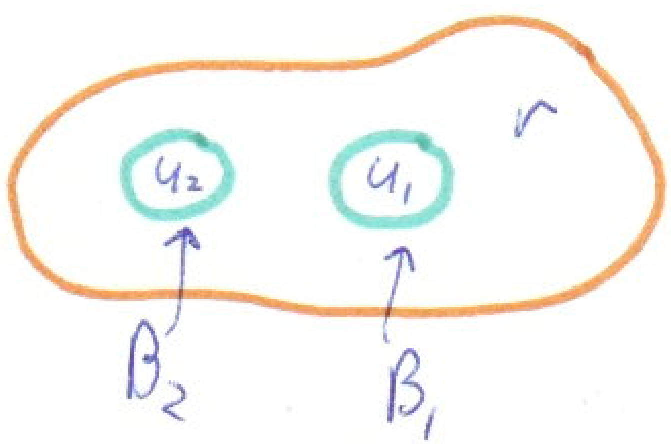
Replace quotient with resolution:

$$\begin{tikzcd}
 & \mathcal{B}_2' \oplus \mathcal{B}_2'' \arrow[d, phantom, "\parallel"] & \mathbb{C}[\{(B,u,r)\}] \arrow[d, phantom, "\parallel"] & \mathbb{C}[\mathcal{C}(M)] \arrow[d, phantom, "\parallel"] \\
\cdots \arrow[r] & \mathcal{B}_2(M,C) \arrow[r, "\partial"] & \mathcal{B}_1(M,C) \arrow[r, "\partial"] & \mathcal{B}_0(M,C)
\end{tikzcd}$$

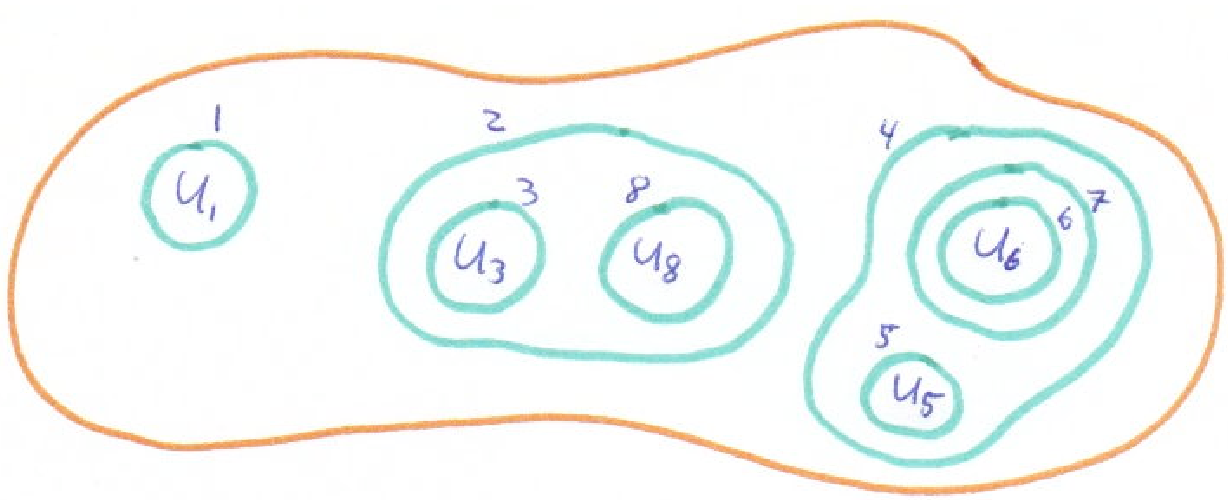
$$(B,u,r) \xmapsto{\partial} u \cdot r$$



$$(B_1, B_2, u_1, r) \stackrel{\partial}{\longmapsto} (B_2, r' \bullet u_1, r'') - (B_1, u_1, r)$$



$$(B_1, B_2, u_1, u_2, r) \stackrel{\partial}{\longmapsto} (B_2, u_2, u_1 \bullet r) - (B_1, u_1, u_2 \bullet r)$$



$\mathcal{B}_k(M, C)$ is defined to be finite linear combinations of k-blob diagrams. A k-blob diagram consists of k blobs (balls) $B_0, \ldots, B_{k-1}$ in M. Each pair B_i and B_j is required to be either disjoint or nested. Each innermost blob B_i is equipped with a null field $u_i \in U$. There is also a C-picture r on the complement of the innermost blobs. The boundary map $\partial : \mathcal{B}_k(M, C) \to \mathcal{B}_{k-1}(M, C)$ is defined to be the alternating sum of forgetting the i-th blob.

- **Relation with TQFTs and skein modules.** $H_0(\mathcal{B}_*(M, C))$ is isomorphic to $A_C(M)$, the dual Hilbert space of the n+1-dimensional TQFT based on C.

- **Relation with Hochschild homology.** When C is a 1-category, $\mathcal{B}_*(S^1, C)$ is homotopy equivalent to the Hochschild complex $\mathrm{Hoch}_*(C)$.

- **Polynomial algebras (possibly truncated) as n categories.** If C is a polynomial algebra viewed as an n-category, then $\mathcal{B}_*(M^n, C)$ is homotopy equivalent to singular chains on a configuration space of M (possibly mod a generalized diagonal).

(see below for details)

- **Functoriality.** The blob complex is functorial with respect to diffeomorphisms. That is, fixing C, the association

$$M \mapsto \mathcal{B}_*(M, C)$$

is a functor from n-manifolds and diffeomorphisms between them to chain complexes and isomorphisms between them.

- **Contractibility for B^n.** The blob complex of the n-ball, $\mathcal{B}_*(B^n, C)$, is quasi-isomorphic to the 1-step complex consisting of n-morphisms of C. (The domain and range of the n-morphisms correspond to the boundary conditions on B^n. Both are suppressed from the notation.) Thus $\mathcal{B}_*(B^n, C)$ can be thought of as a free resolution of C.

- **Disjoint union.** There is a natural isomorphism

$$\mathcal{B}_*(M_1 \sqcup M_2, C) \cong \mathcal{B}_*(M_1, C) \otimes \mathcal{B}_*(M_2, C).$$

- **Gluing.** Let M_1 and M_2 be n-manifolds, with Y a codimension-0 submanifold of ∂M_1 and $-Y$ a codimension-0 submanifold of ∂M_2. Then there is a chain map

$$\mathrm{gl}_Y : \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \to \mathcal{B}_*(M_1 \cup_Y M_2).$$

- **Evaluation map.** There is an 'evaluation' chain map

$$\mathrm{ev}_M : C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \to \mathcal{B}_*(M).$$

(Here $C_*(\mathrm{Diff}(M))$ is the singular chain complex of the space of diffeomorphisms of M, fixed on ∂M.)

Restricted to $C_0(\mathrm{Diff}(M))$ this is just the action of diffeomorphisms described above. Further, for any codimension-1 submanifold $Y \subset M$ dividing M into $M_1 \cup_Y M_2$, the following diagram (using the gluing maps described above) commutes.

$$\begin{tikzcd}
C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \arrow[r, "\mathrm{ev}_M"] & \mathcal{B}_*(M) \\
C_*(\mathrm{Diff}(M)) \otimes C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y^{\mathrm{Diff}} \otimes \mathrm{gl}_Y"] \arrow[r, bend right, "\mathrm{ev}_{M_1} \otimes \mathrm{ev}_{M_2}"'] & \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y"']
\end{tikzcd}$$

In fact, up to homotopy the evaluation maps are uniquely characterized by these two properties.

Lemma. Let $f : P^k \times M \to M$ be a k a k-parameter family of diffeomorphisms and $\{U_i\}$ be an open cover of M. Then f is homotopic in $C_k(\mathrm{Diff}(M))$ to $\sum f_j$, where is each f_j is supported on a union of at most k of the U_i's. (This is, if $f_j : Q^k \times M \to M$, then $f(q, x) = f(q', x)$ for all q, q' unless x is in the aforementioned union of U_i's.)

- **A_∞ categories for $n{-}1$-manifolds.** For Y an $n{-}1$-manifold, the blob complex $\mathcal{B}_*(Y \times I, C)$ has the structure of an A_∞ category. The multiplication (m_2) is given my stacking copies of the cylinder $Y \times I$ together. The higher m_i's are obtained by applying the evaluation map to $i{-}2$-dimensional families of diffeomorphisms in $\mathrm{Diff}(I) \subset \mathrm{Diff}(Y \times I)$. Furthermore, $\mathcal{B}_*(M, C)$ affords a representation of the A_∞ category $\mathcal{B}_*(\partial M \times I, C)$.

- **Gluing formula.** Let $Y \subset M$ divide M into manifolds M_1 and M_2. Let $A(Y)$ be the A_∞ category $\mathcal{B}_*(Y \times I, C)$. Then $\mathcal{B}_*(M_1, C)$ affords a right representation of $A(Y)$, $\mathcal{B}_*(M_2, C)$ affords a left representation of $A(Y)$, and $\mathcal{B}_*(M, C)$ is homotopy equivalent to $\mathcal{B}_*(M_1, C) \otimes_{A(Y)} \mathcal{B}_*(M_2, C)$.

(More generally, can define an A_∞ k-category for $n{-}k$-manifolds, and prove a similar gluing theorem.)

There is a version of the blob complex for C an A_∞ n-category. If C is the A_∞ n-category based on maps of $B^0, B^1, \ldots B^n \to W$, then $\mathcal{B}_*(M, C)$ is homotopy equivalent to $C_*(\{\text{maps } M \to W\})$.

In place of an exact triangle, $A_{\mathrm{Kh}}(W^4, L)$ has a collapsing spectral sequence.

The blob complex and configuration spaces:

$$
\begin{aligned}
C = \mathbb{C}[t] \quad &\Longrightarrow \quad \mathcal{B}_*(M, C) \simeq C_*(\Sigma^\infty(M)) \\
C = \mathbb{C}[t]/(t^k) \quad &\Longrightarrow \quad \mathcal{B}_*(M, C) \simeq C_*(\Sigma^\infty(M), \Delta_k) \\
C = \mathbb{C}[t_1, \ldots, t_m] \quad &\Longrightarrow \quad \mathcal{B}_*(M, C) \simeq C_*(\Sigma^\infty_m(M))
\end{aligned}
$$

Compatibly with the action of $C_*(\mathrm{Diff}(M))$.