From $(n+\varepsilon)$-dimensional TQFTs to $(n+1)$-dimensional TQFTs

(without semisimplicity assumptions)

these slides: https://canyon23.net/math/talks/np1.pdf

joint work with David Reutter

{oriented, unoriented, spin, ...}

$n+\varepsilon$-dimensional ^ TQFTs are plentiful and easy to construct

- ingredient: functors $\mathcal{C}^k : \left(k\text{-balls}, \begin{Bmatrix} \text{diffeo} \\ \text{homeo} \end{Bmatrix}\text{-morphisms} \right) \longrightarrow \underline{\text{Set}}$

  ($0 \leq k \leq n$; perhaps linearized when $k=n$)

- examples: ① $\mathcal{C}(X) := \{ \text{maps } X \to T \}$ e.g. $T = BG$

  ② $\mathcal{C}(X) := \{ C\text{-string diagrams on } X \}$

  C: H-pivotal n-category, $H = \text{Spin}, SO, \ldots$

- require $\{\mathcal{C}^k\}$ to be compatible with gluing, restrictions to boundary

* Extend $\mathcal{C}$ from balls to manifolds (of dimension $0 \dots n$) via colimit construction (over poset of ball decompositions)

$\mathcal{C}(M^n)$: "skein module" (vector space)

$\mathcal{C}(Y^{n-1})$: objects of cylinder category

$\mathcal{C}(Y^{n-1} \times I)$: morphisms of cylinder category

linear 1-category

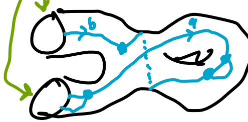
fully extended → ⋮

$(n-k)$-category $A(X^k)$, j-morphisms $= \mathcal{C}(X \times B^j)$

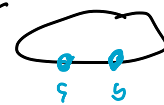
Example $n=2$

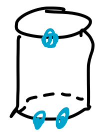
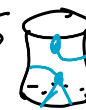
∂-condition

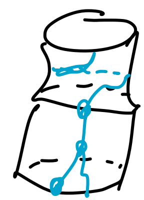
c

$A(M^2;c) = \mathbb{K}[\{$  $\}]/\sim$

$A(S^1) = \Big[$

obj: $\{$  $\}$

mor: $A($  $) = \mathbb{K}[\{$  $\}]/\sim$

composition:  ← stacking cylinders

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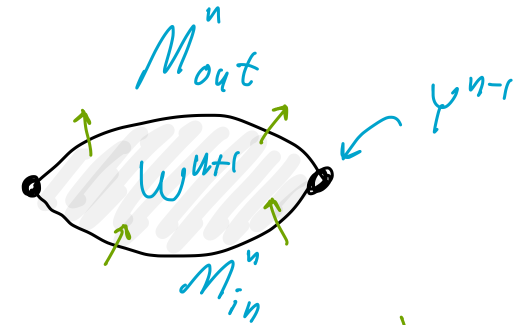
$A(M^n;c)\ ,\ c \in \mathcal{C}(\partial M)$

Note: $\{A(M;\bullet)\}$ affords an action of the 1-cat $A(\partial M)$

from $n+\varepsilon$ to $n+1$

- for every bordism



$\partial W = M_{out} \cup_Y \overline{M_{in}}$

$\partial M_{out} = Y = \partial M_{in}$

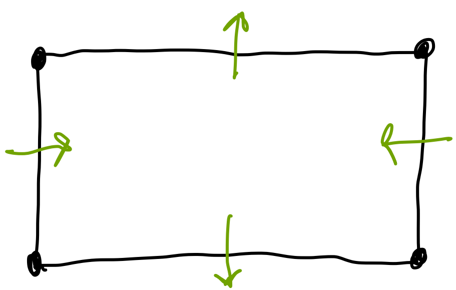
want: $Z(W^{n+1}) : A(M_{in}) \to A(M_{out})$

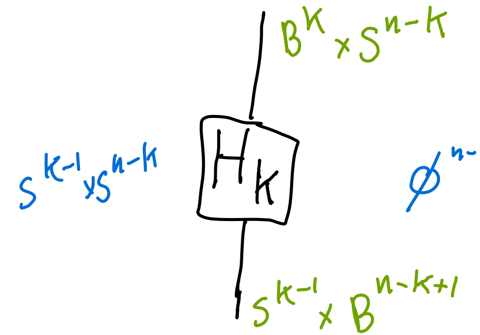
$A(Y)$-module map

compatible with gluing of $(n+1)$-manifolds and homeomorphisms of n-manifolds

k-handle bordism

$$H_k = B^k \times B^{n+1-k} : S^{k-1} \times B^{n-k+1} \to B^k \times S^{n-k}$$





Std. topological fact:

$$\left[(n+1)\text{-manifolds}\right] \overset{\sim}{=} \left[\text{handle decompositions}\right] \Big/ \sim$$

1. distant reordering
2. isotopy of attachment
3. handle cancellations

Strategy:

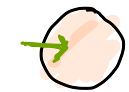
Define $Z(H_0)$, $Z(H_1)$, ...

... $Z(H_{n+1})$. Then verify 3.

(1 and 2 come for free.)

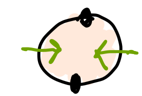
Warm-up: $n=1$ case

- Choose $Z(D^2): A(S^1) \to k$ $\qquad H_0$ (0-handle)



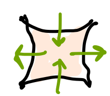
$\rightsquigarrow$ Pairing $P_0 : A(B^1) \otimes A(B^1) \to k$

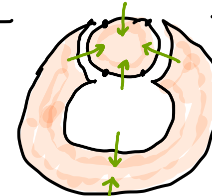
$$x \otimes y \longmapsto Z(B^2)(x \cup y)$$



$\rightsquigarrow$ If P_0 is non-degenerate, $\exists$ copairing

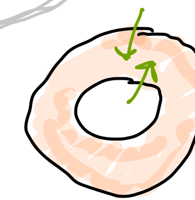
$$Q_0 : {}_1A(B^1)_2 \otimes {}_3A(B^1)_4 \to {}_1A(B^1)_4 \otimes {}_3A(B^1)_2$$

$\rightsquigarrow$ Define $Z(H_1) = Z($  $) = Q_0$

$\rightsquigarrow$ Compute $Z(S^1 \times I) = Z\Big[$  $\Big] = Z(H_0) \bullet Z(H_1)$

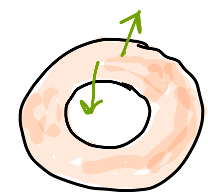
$= Z(H_0) \bullet Q_0$

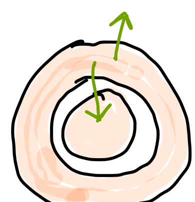
$\rightsquigarrow$ Pairing $P_1 = Z(S^1 \times I) : A(S^1) \otimes A(S^1) \to \mathbb{K}$



$\rightsquigarrow$ If P_1 is non-degenerate, $\exists$ copairing

$$Q_1 : \mathbb{K} \to A(S^1) \otimes A(S^1)$$



$\rightsquigarrow$ compute $Z(H_2) = Z\Big[$  $\Big] = Z(H_0) \bullet Q_1$

Conclusion: If extension from $1+\varepsilon$ to $1+1$ exists, then it is completely determined by choice of $Z(H_0): A(S^1) \to \mathbb{K}$. A necessary condition for $Z(H_0)$ to extend is that the pairings P_0 and P_1 are non-degenerate.

Thm (W, Reutter). Let $A(\cdot)$ be an $n+\varepsilon$-dimensional TQFT as above. Choose $Z(B^{n+1}) = Z(H_0): A(S^n) \to \mathbb{K}$.

Then $Z(\cdots)$ extends to a full $n+1$-dim'l TQFT if and only if the inductively defined pairings

$$p_k : A(S^k \times B^{n-k}) \otimes A(S^k \times B^{n-k}) \to \mathbb{K} \quad , \quad 0 \le k \le n-1$$

are non-degenerate.

Remark 1: If $p_0, p_1, \dots, p_m$ are non-degenerate, then can define $Z(\cdots)$ on $n+1$-dim'l handlebodies, all handles of index $\le m+1$, invariant under handle cancellations of index $\le m+1$.

Remark 2: Proof depends only 2-functor

A: {n-1-manifolds, n-manifolds, homeomorphisms} $\longrightarrow$ target 2-cat

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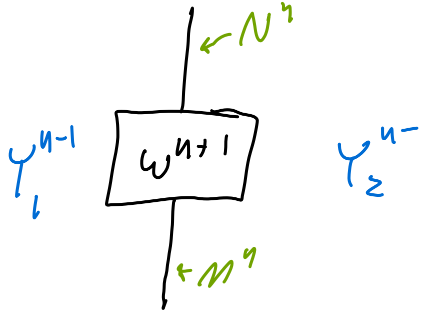
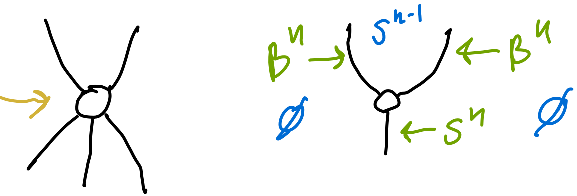
Earlier results:

- W. 2006 — semisimple, positive def. case
- Lurie 2009 — ∞-cat case, framed manifolds, non-pivotal n-cats

Proof:

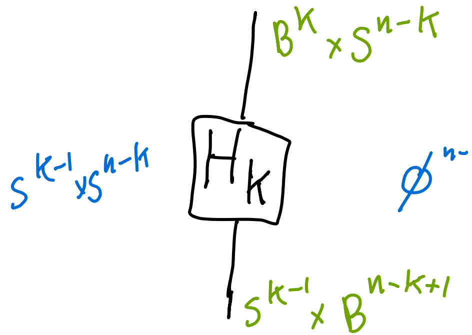


[All diagrams are in target 2-cat, but most labels are in source bordism 2-cat.]

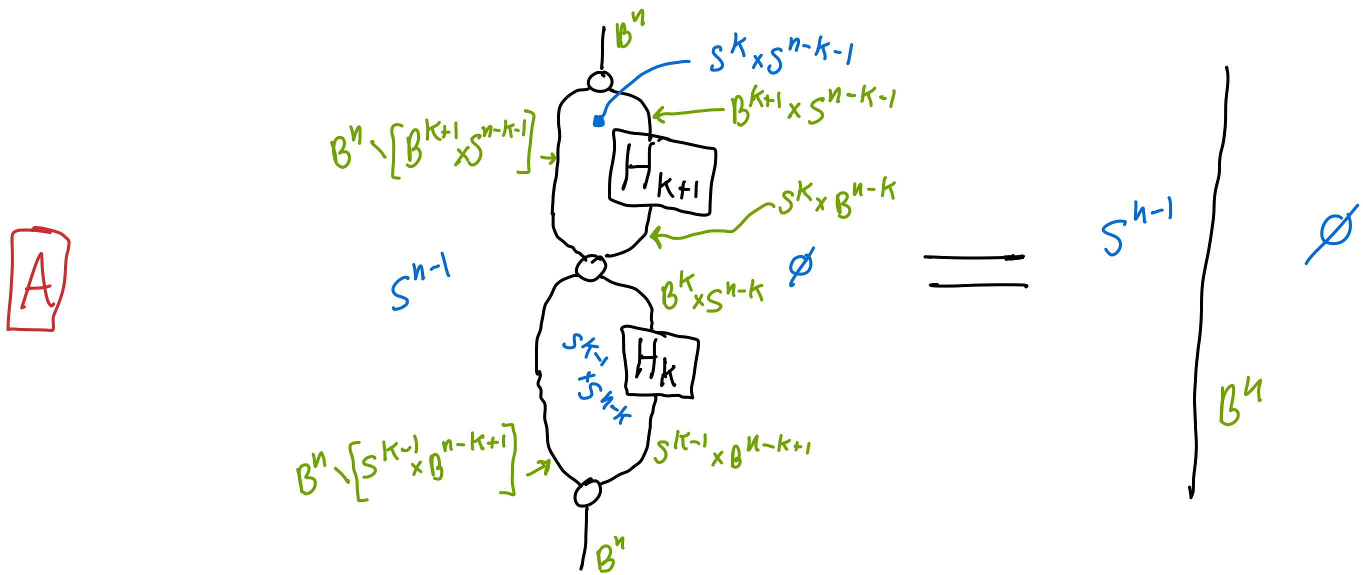


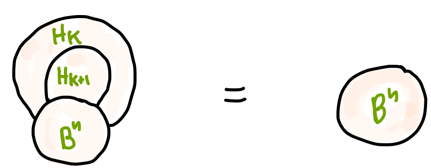
homeomorphism from $n+\varepsilon$-dim'l TQFT

We want to define $Z(H_K)$

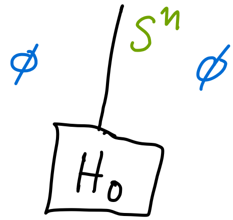


Satisfying handle cancellation:





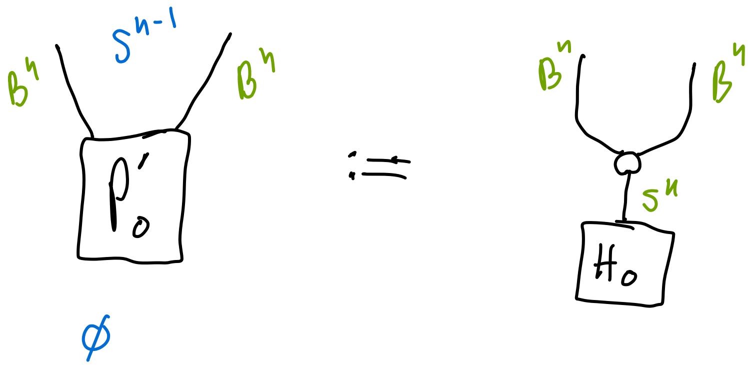
Assume $\exists\ Z(H_0)$



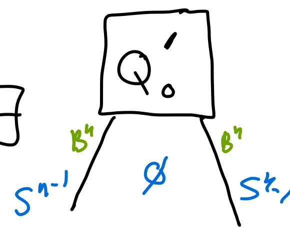
$H_0 = B^{n+1}$

such that the induced pairings P_0'

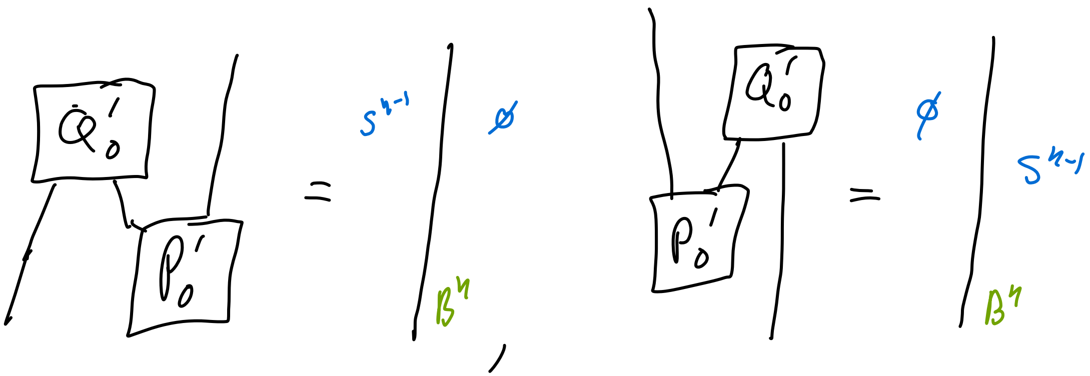
$\boxed{B}$



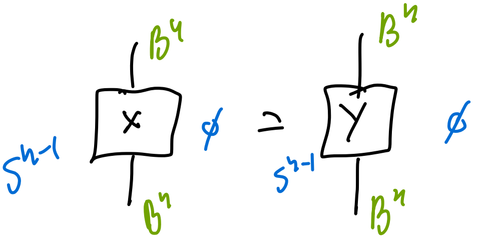
is non-degenerate, in the sense that $\exists$

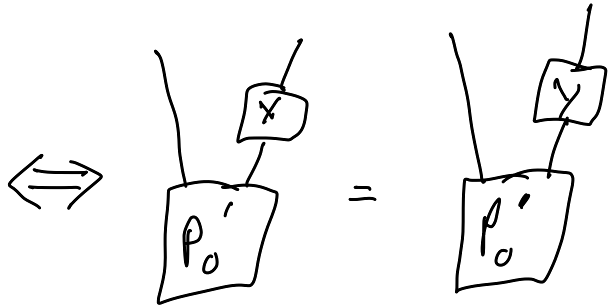


$\boxed{C}$



Lemma. D

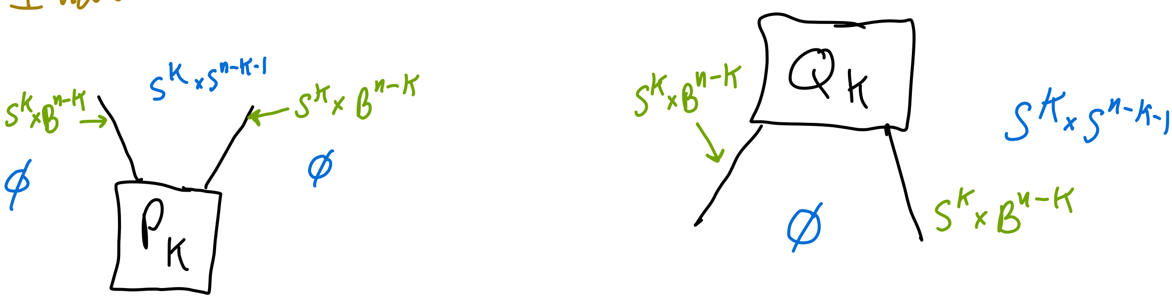




Pf. Easy.

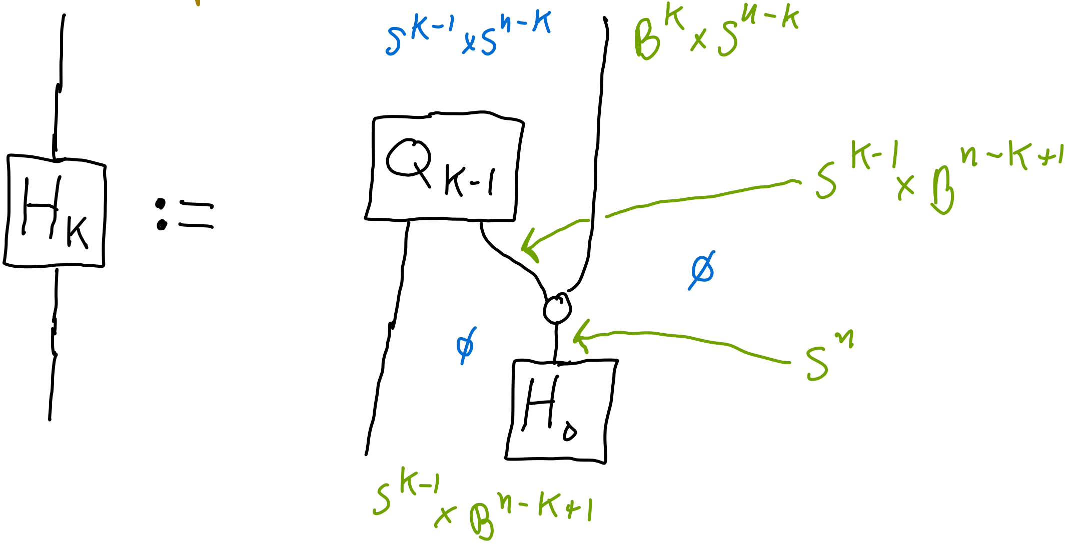
Define $P_0 = P_0' \amalg P_0'$, $Q_0 = Q_0' \amalg Q_0'$

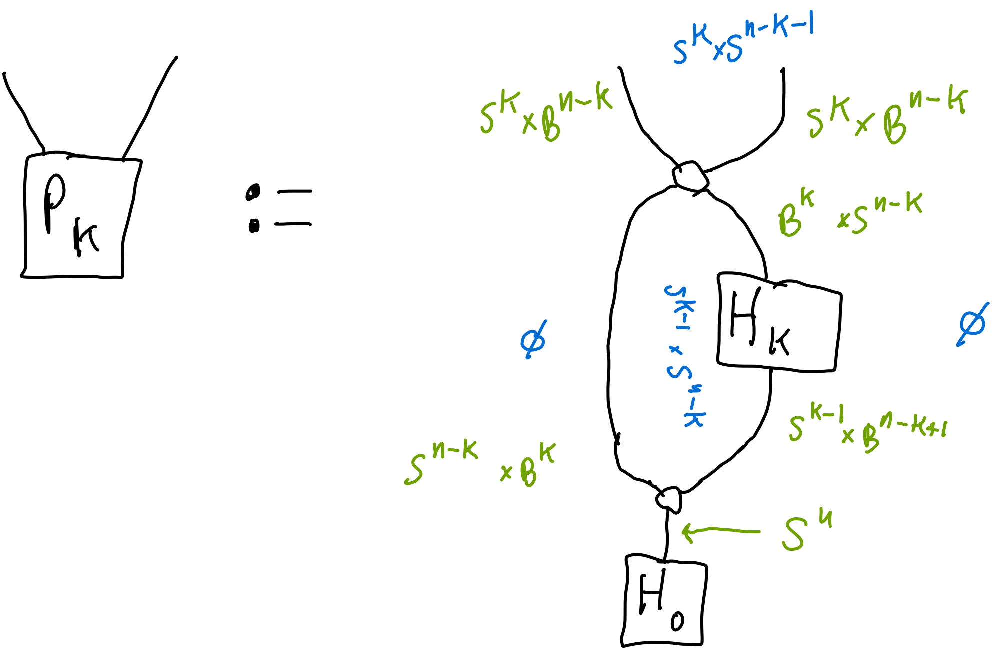
Inductive assumptions:



satisfying two zig-zags

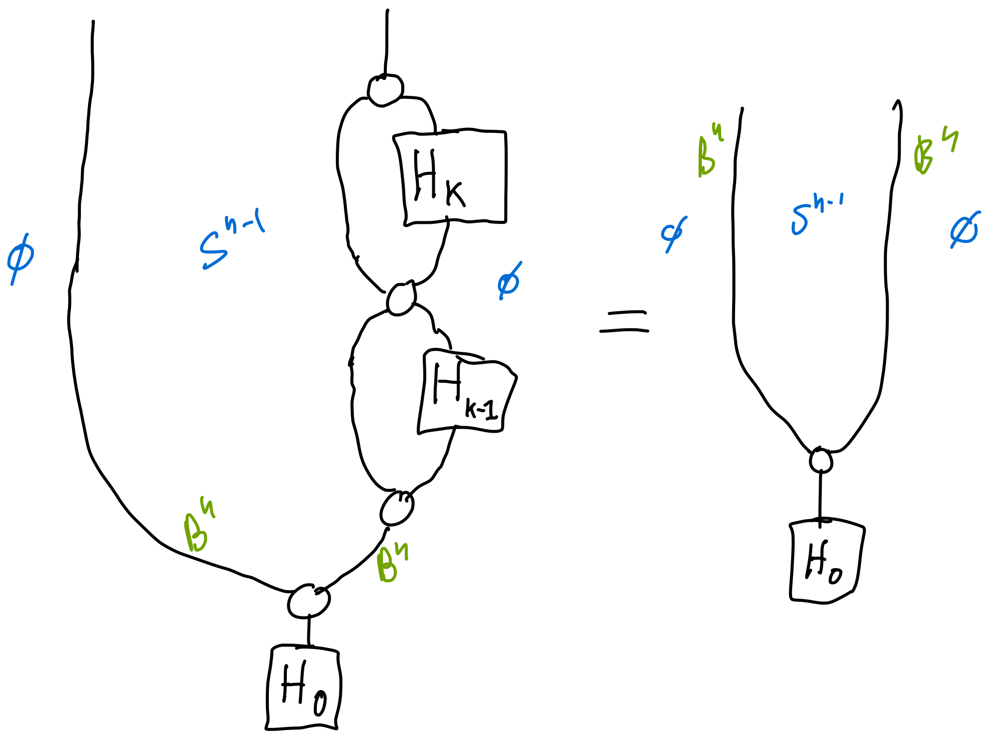
Inductive steps....



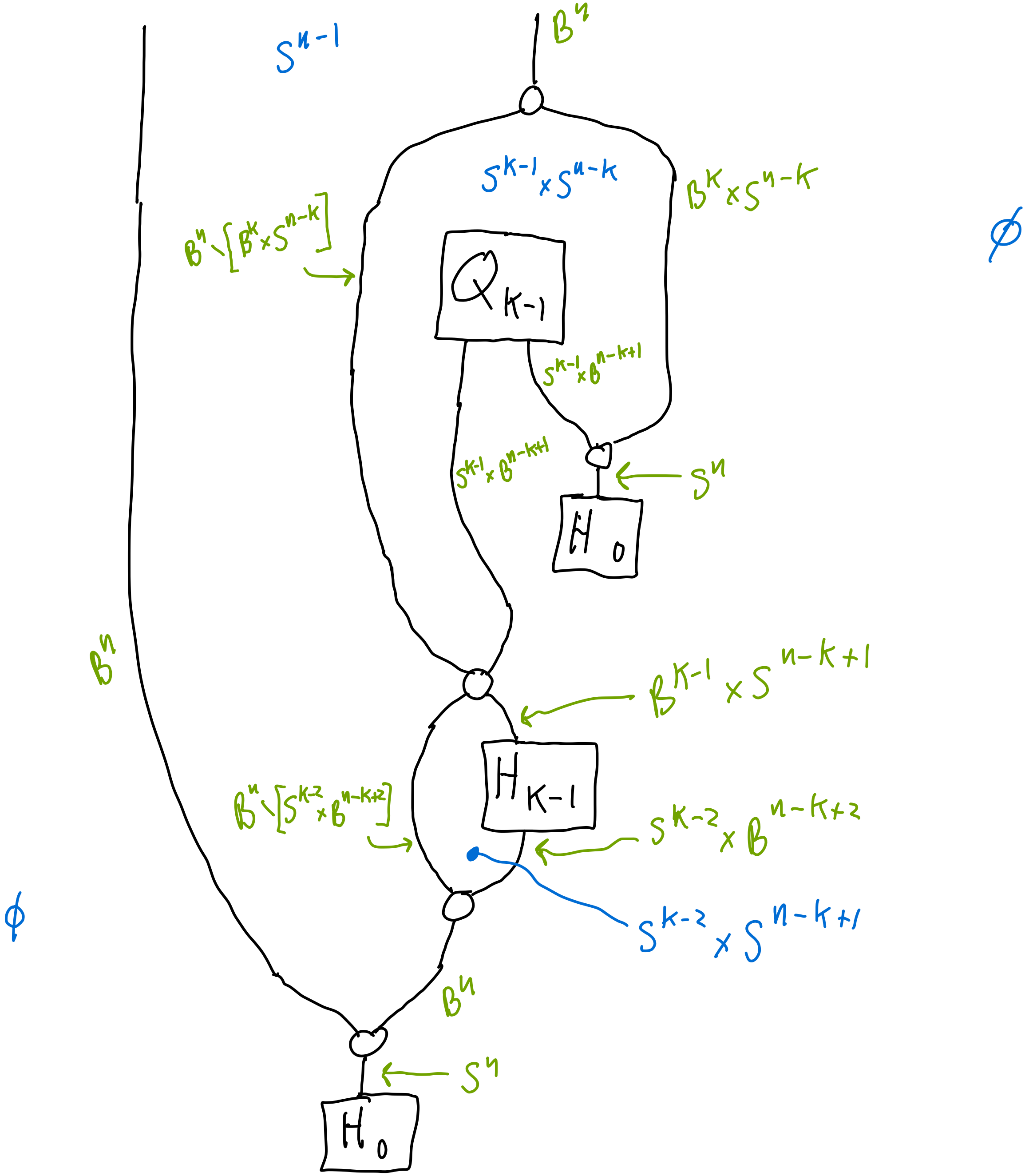


:= copairing of P_K, if it exists

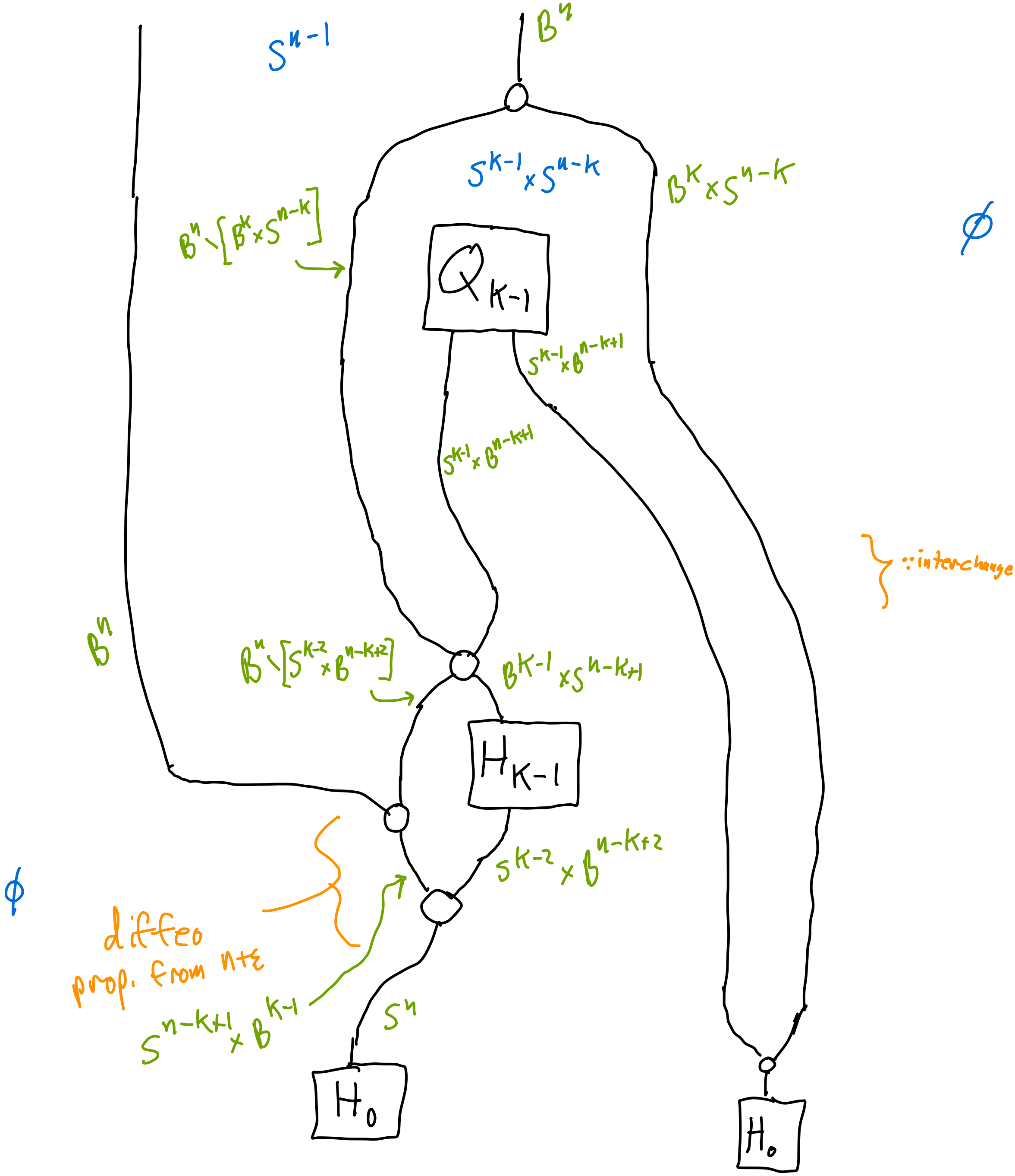
handle cancellation. By B + D, A follows from

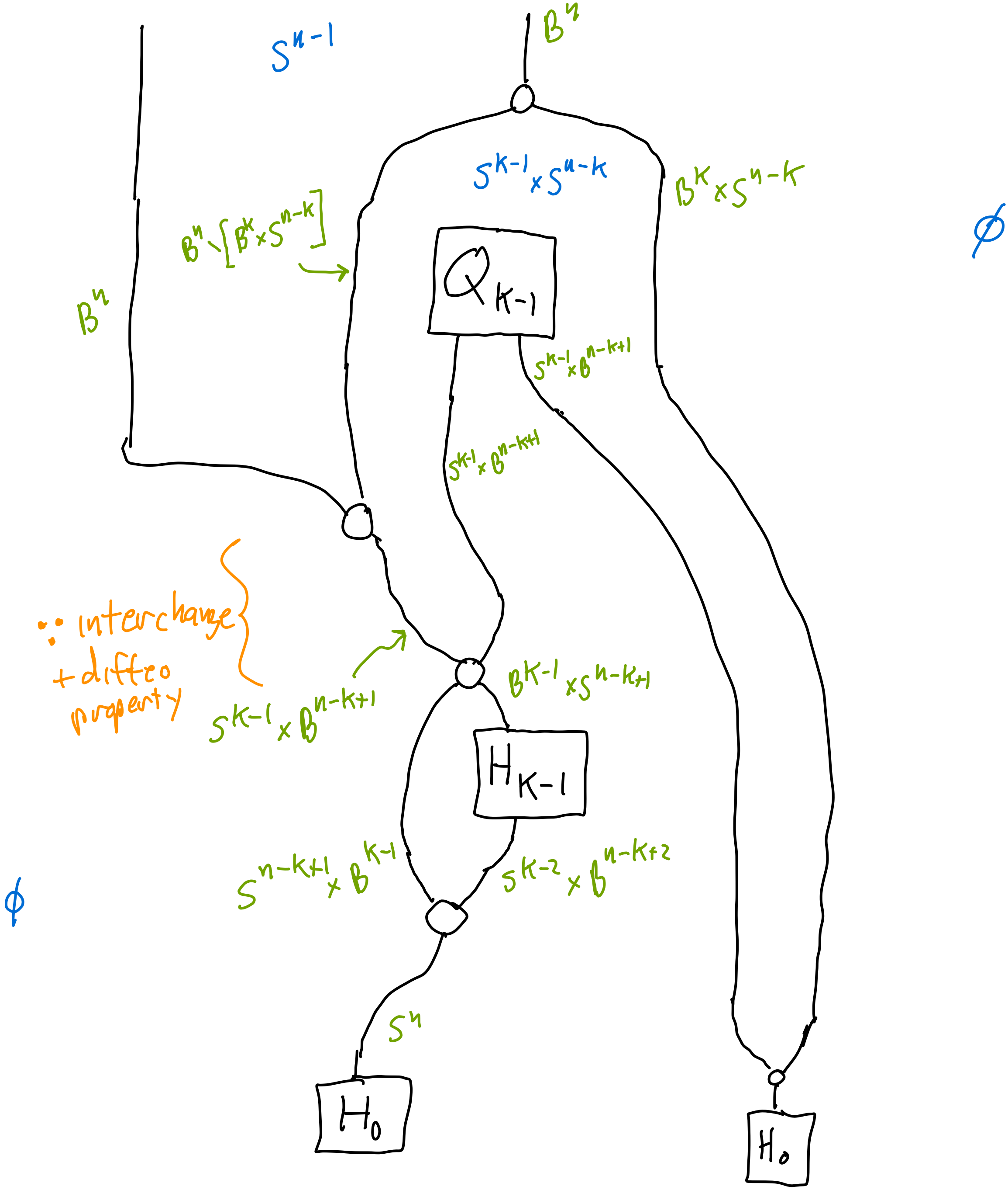


expand H_K

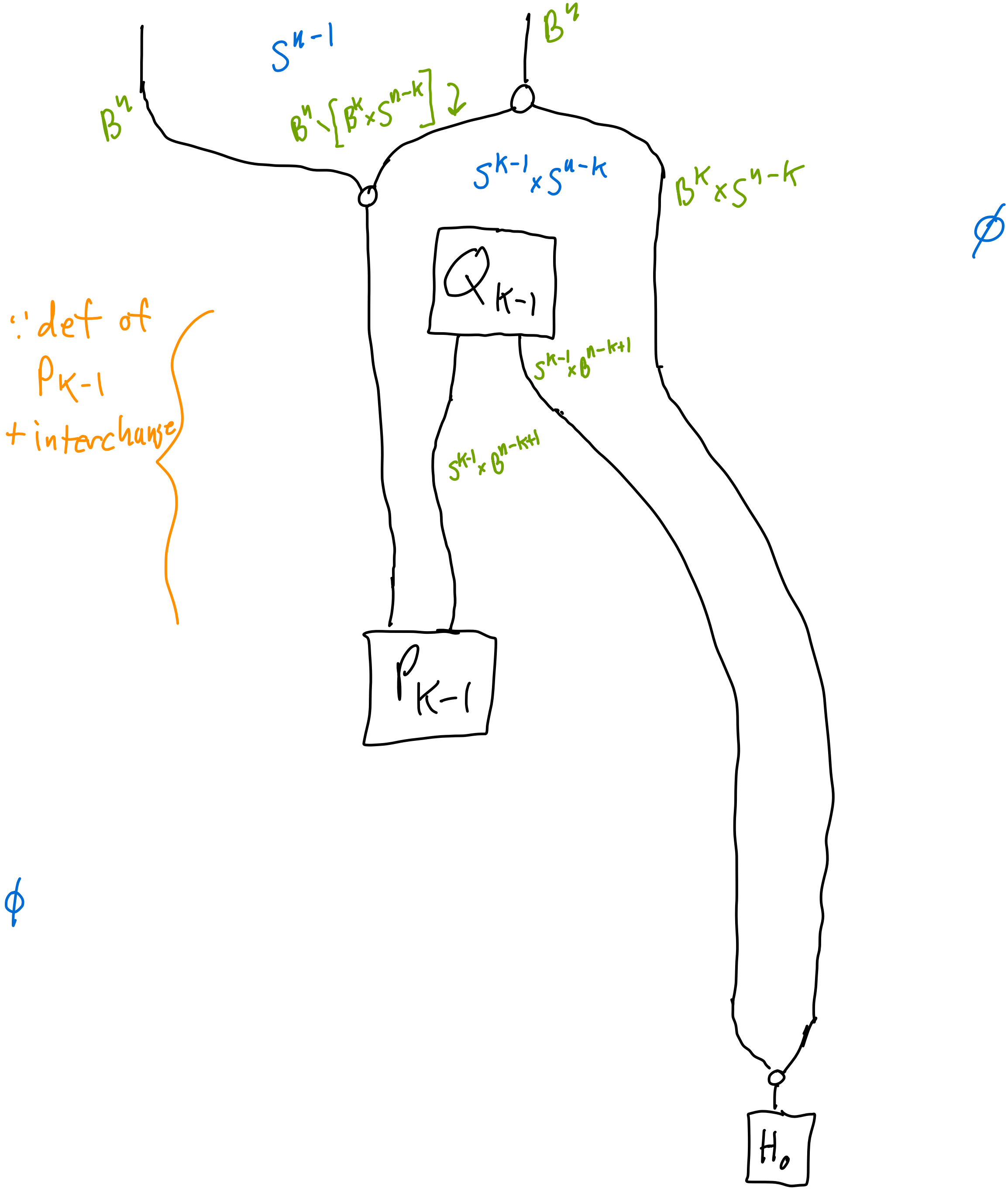


S^{n-1}
B^n
$S^{k-1} \times S^{n-k}$
$B^k \times S^{n-k}$
$\emptyset$
$B^n \setminus [B^k \times S^{n-k}]$
Q_{k-1}
$S^{k-1} \times B^{n-k+1}$
$S^{k-1} \times B^{n-k+1}$
S^n
H_0
B^n
$B^{k-1} \times S^{n-k+1}$
H_{k-1}
$B^n \setminus [S^{k-2} \times B^{n-k+2}]$
$S^{k-2} \times B^{n-k+2}$
$\emptyset$
$S^{k-2} \times S^{n-k+1}$
B^n
S^n
H_0

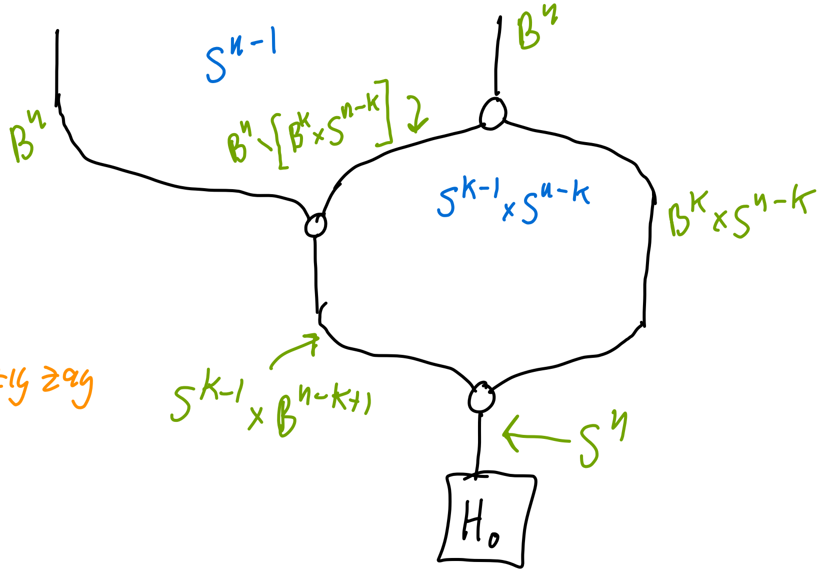

S^{n-1}
B^n
$S^{k-1}\times S^{n-k}$
$B^n\setminus[B^k\times S^{n-k}]$
$B^k\times S^{n-k}$
$\emptyset$
Q_{k-1}
$S^{k-1}\times B^{n-k+1}$
$S^{k-1}\times B^{n-k+1}$
:: interchange
B^n
$B^n\setminus[S^{k-2}\times B^{n-k+2}]$
$B^{k-1}\times S^{n-k+1}$
H_{k-1}
$S^{k-2}\times B^{n-k+2}$
$\emptyset$
diffeo
prop. from $n+\varepsilon$
$S^{n-k+1}\times B^{k-1}$
S^n
H_0
H_0



S^{n-1}
B^n
S^{k-1} × S^{n-k}
B^k × S^{n-k}
B^n \ [B^k × S^{n-k}]
Q_{k-1}
ϕ
B^n
S^{k-1} × B^{n-k+1}
S^{k-1} × B^{n-k+1}
∵ interchange + diffeo property
B^{k-1} × S^{n-k+1}
S^{k-1} × B^{n-k+1}
H_{k-1}
S^{n-k+1} × B^{k-1}
S^{k-2} × B^{n-k+2}
ϕ
S^n
H_0
H_0

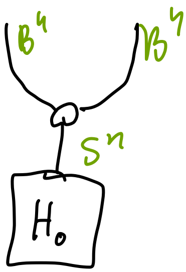


∵ def of P_{k-1} + interchange



∵ P-Q zigzag

=



by diffeo property

QED

Mysterious (to me) non-semisimple TQFT constructions:

1990s: Lyubashenko, Kuperberg, Hennings, ...

RT-ish:

arXiv:1912.02063v2

3-DIMENSIONAL TQFTS FROM NON-SEMISIMPLE MODULAR CATEGORIES

MARCO DE RENZI, AZAT M. GAINUTDINOV, NATHAN GEER, BERTRAND PATUREAU-MIRAND, AND INGO RUNKEL

Abstract. We use modified traces to renormalize Lyubashenko's closed 3-manifold invariants coming from twist non-degenerate finite unimodular ribbon categories. Our construction produces new topological invariants which

TV-ish:

arXiv:1809.07991v2

KUPERBERG AND TURAEV-VIRO INVARIANTS IN UNIMODULAR CATEGORIES

FRANCESCO COSTANTINO, NATHAN GEER, BERTRAND PATUREAU-MIRAND, AND VLADIMIR TURAEV

Abstract. We give a categorical setting in which Penrose graphical calculus naturally extends to graphs drawn on the boundary of a handlebody. We use it to introduce invariants of 3-manifolds presented by Heegaard splittings. We recover Kuperberg invariants when the category arises from

$\rightsquigarrow$ consider non-$\otimes$-unital $\otimes$-categories.

$\rightsquigarrow$ non-$\otimes$-unital skein theory [D. Jordan]

(this is still work in progress)

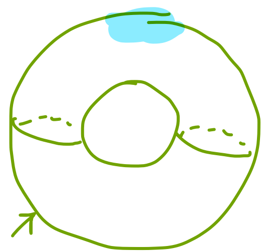
$n=3$, $\mathcal{P}$ = non-$\otimes$-unital ribbon category (e.g. projective ideal in....)

$$A(M^3) := \mathbb{K}\left[\{\mathcal{P}\text{-ribbon-graphs in } M\}\right] \Big/ \langle \text{local relations with non-empty } \partial\text{-condition} \rangle$$

Then

$A(S^3)^* \longleftrightarrow$ space of "modified traces"

$Z(H_0) \in A(S^3)^* \longleftrightarrow$ choice of modified trace pairing P_0 non-degenerate $\longleftrightarrow$ modified trace non-degenerate

$A\Big[$  $\Big]$ $\longleftrightarrow$ coend of

$S^1 \times D^2$

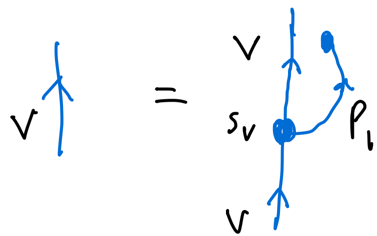
$X \otimes Y \mapsto \mathrm{mor}_P(X^* \otimes Y, -)$

$\cap$

$\mathrm{Rep}(P)$

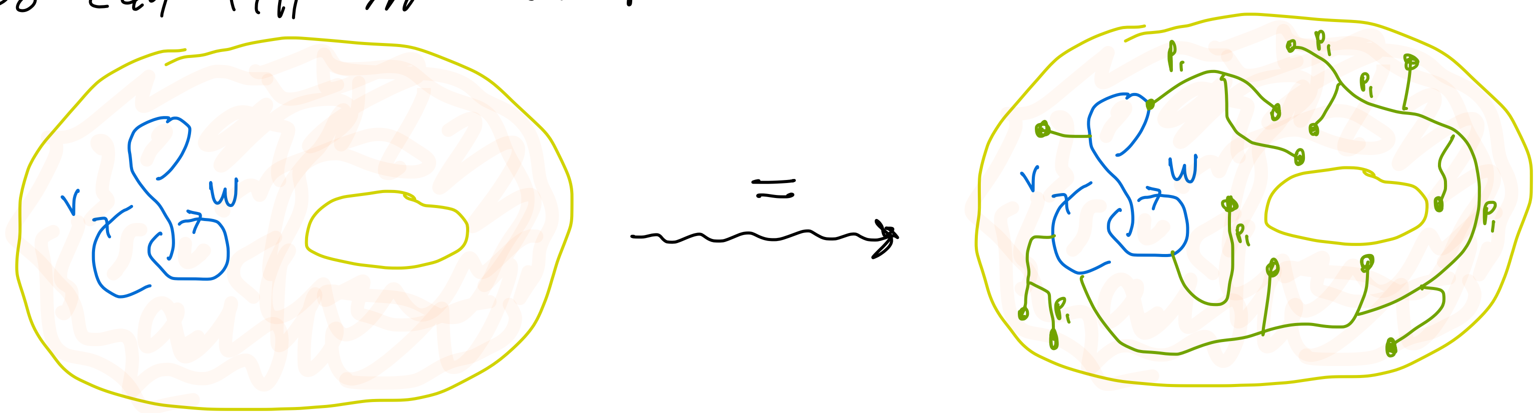
$Z(H_2) : A(S^1 \times D^2) \to A(D^2 \times S^1) \rightsquigarrow$ integral for this coend

Note: In Geer et al examples, P_1 (projective cover of $\otimes$-unit) has the following "weak $\otimes$-unit" property:



$\forall$ objects $V \in \mathcal{P}$

So can fill M^3 with "tendrils":



Non-semisimple Crane-Yetter 3+1-dim'l TQFT

$\mathcal{P}$ as above. W^4 oriented 4-manifold. G: $\mathcal{P}$-ribbon graph in ∂W. Goal: evaluate $Z(W)(G) \in \mathbb{k}$.

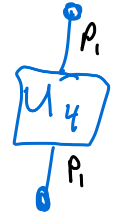
Choose handle decomposition $W_0 \subset W_1 \subset W_2 \subset W_3 \subset W_4 = W$

(W_i = 0-handles $\cup \cdots \cup$ i-handles)

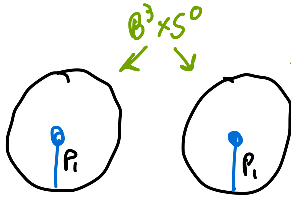
Recall from above

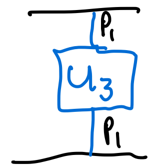
$$Z(H_i): A(B^i \times S^{3-i}) \to A(S^{i-1} \times B^{4-i})$$

Define

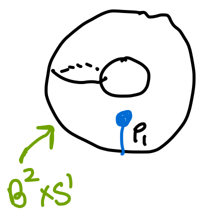
$Z(H_4)(\emptyset) =$ 

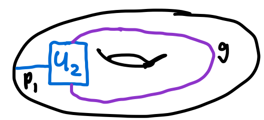
$\in A(S^3)$

$Z(H_3)\Big[$ 

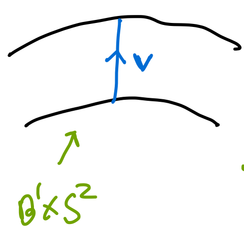
$\Big] =$ 

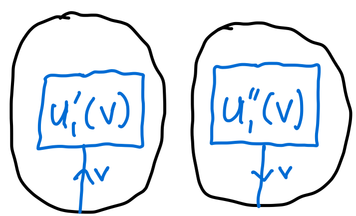
$\in A(S^2 \times B^1)$

$Z(H_2)\Big[$ 

$\Big] =$ 

$\in A(S^1 \times B^2)$

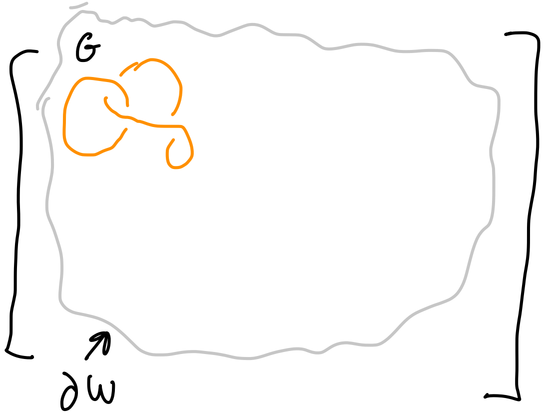
$Z(H_1)\Big[$ 

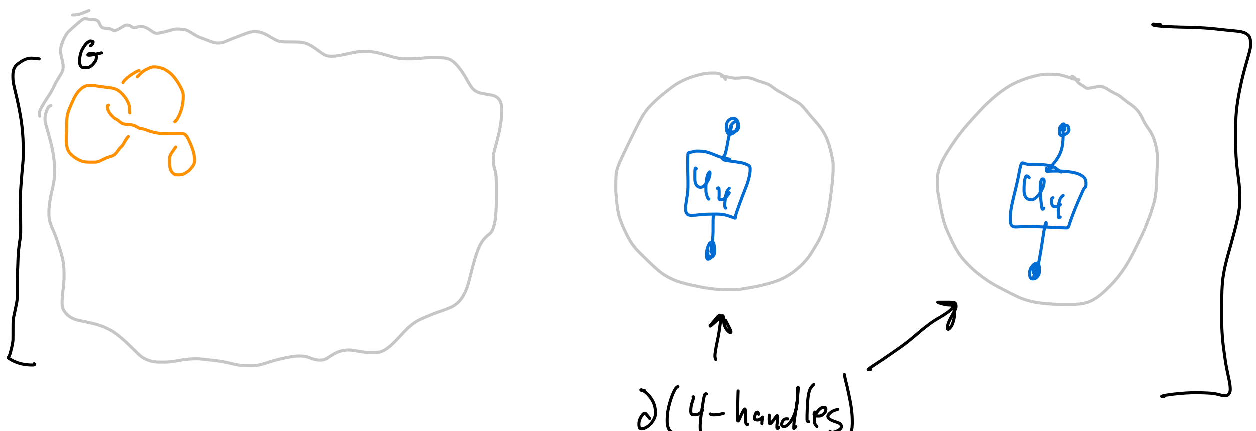
$\Big] =$ 

$\in A(S^0 \times B^3)$

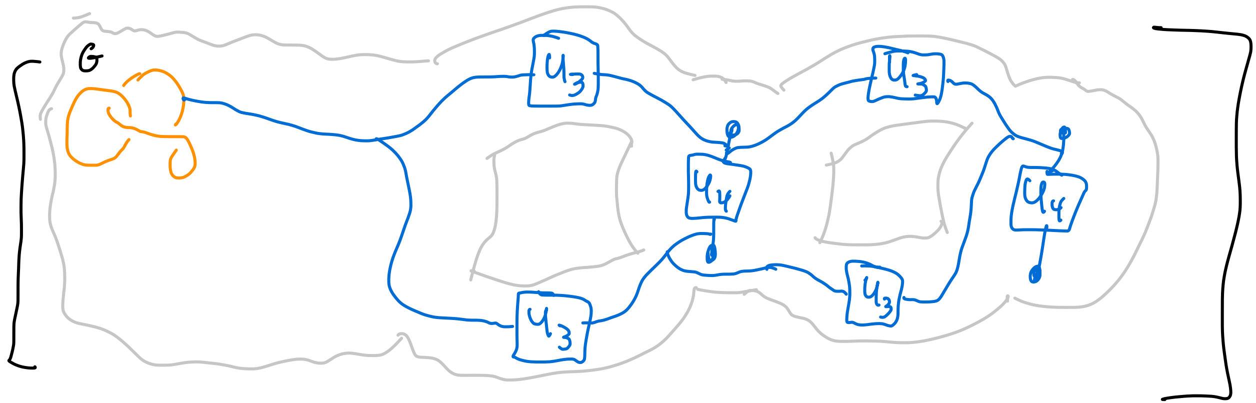
$$Z(H_0) = \mathrm{mtr} : \mathcal{A}(S^3) \to \Bbbk$$

Then....

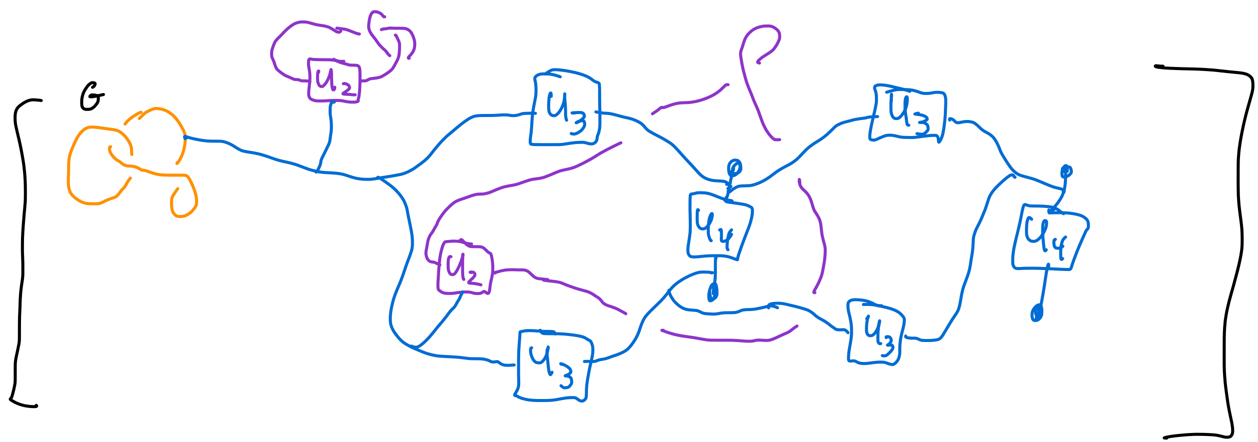
$Z(W_4)$ 

$= Z(W_3)$ 

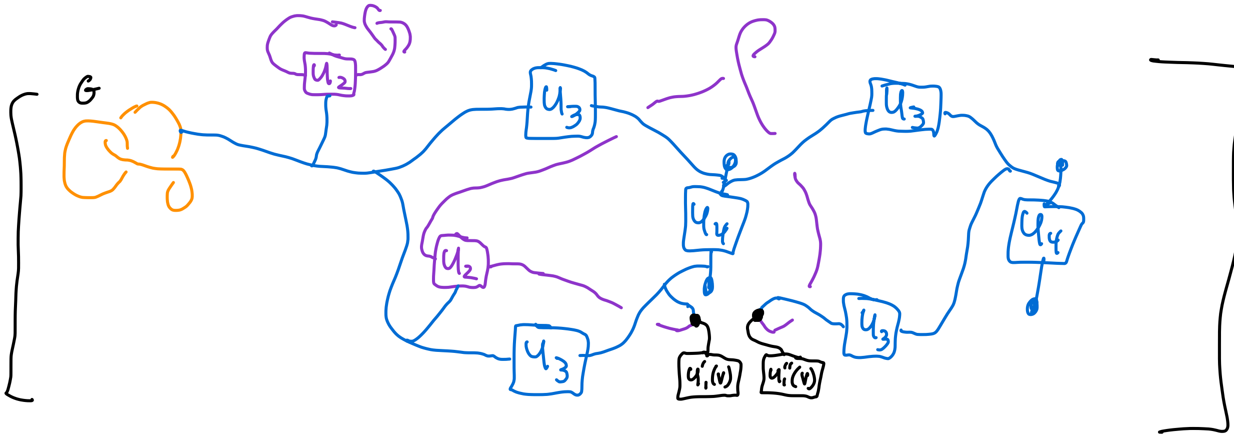
$= Z(W_2)$



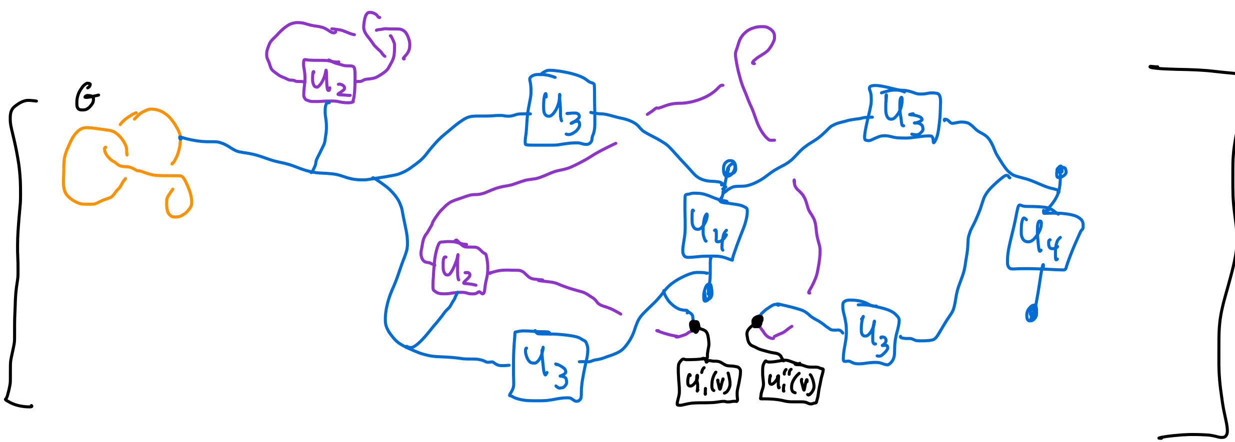
$= Z(W_1)$



$= Z(w_0)$



$= \mathrm{mtr}$



Thm (W, Reutter). Let $A(\cdot)$ be an $n+\varepsilon$-dimensional TQFT as above. Choose $Z(B^{n+1}) = Z(H_0): A(S^n) \to \mathbb{K}$.

Then $Z(\cdots)$ extends to a full $n+1$-dim'l TQFT if and only if the inductively defined pairings

$$P_K: A(S^K \times B^{n-K}) \otimes A(S^K \times B^{n-K}) \to \mathbb{K}, \quad 0 \leq K \leq n-1$$

are non-degenerate.

Remark 1: If $P_0, P_1, \ldots, P_m$ are non-degenerate, then can define $Z(\cdots)$ on $n+1$-dim'l handlebodies, all handles of index $\leq m+1$, invariant under handle cancellations of index $\leq m+1$.

- very common for P_0 to be non-generate:
  - $n=2$ or $n=3$, $\mathrm{Rep}_q(\mathfrak{g})$, q generic. $\Rightarrow$ can define generalized Jone polynomials for links in $\partial(S^1\times B^3 \natural \cdots \natural S^1\times B^3)$

- $\{(n+1)\text{-dim'l } k\text{-handle bodies}\}/(\leq k)\text{-handle moves}$

  $\simeq$ $(n+1)$-dim'l manifolds w/ $(\leq k)$-handle structure

  except when $(n+1, k) = (4, 2)$.

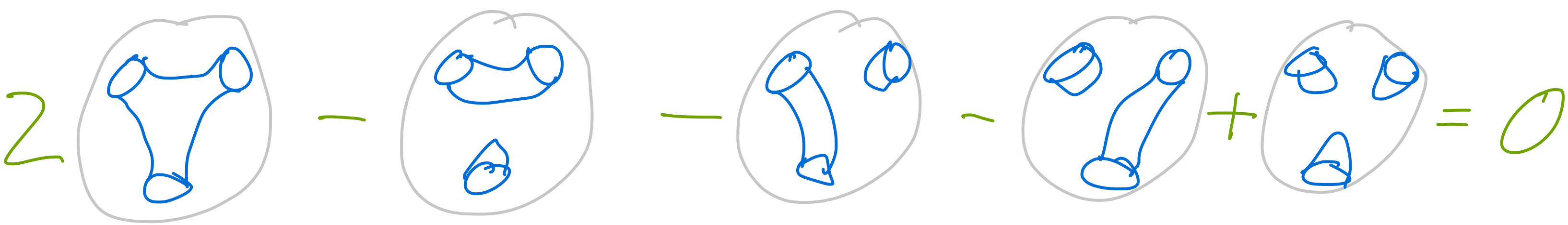
  (related to Andrews–Curtis problem)

• interesting $(n+1, k) = (4,2)$ example:

$$A(M^3) := \Bbbk\left[\{\text{unoriented surfaces in } M\}\right] / \sim$$

① partition relations

②



In this example, P_0 <u>and</u> P_1 are non-degenerate, but P_2 is degenerate.

arXiv:1912.02063v2

3-DIMENSIONAL TQFTS FROM NON-SEMISIMPLE MODULAR CATEGORIES

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Kuperberg
Hennings
⋮

Abstract. We use modified traces to renormalize Lyubashenko's closed 3-manifold invariants coming from twist non-degenerate finite unimodular ribbon categories. Our construction produces new topological invariants which

⋮

A *trace* t *on a tensor ideal* $\mathscr{I} \subset \mathscr{C}$ is a family of linear maps

$$\{\mathrm{t}_X : \mathrm{End}_{\mathscr{C}}(X) \to \Bbbk\}_{X \in \mathscr{I}}$$

subject to the following conditions:

1) *Cyclicity*: For all $X, Y \in \mathscr{I}$ and $f : X \to Y$, $g : Y \to X$ we have

$$\mathrm{t}_Y(f \circ g) = \mathrm{t}_X(g \circ f);$$

2R) *Right partial trace*: For all $X \in \mathscr{I}$, $V \in \mathscr{C}$ and $h \in \mathrm{End}_{\mathscr{C}}(X \otimes V)$,

$$\mathrm{t}_{X \otimes V}(h) = \mathrm{t}_X(\mathrm{tr}_{\mathrm{R}}(h));$$

2L) *Left partial trace*: For all $X \in \mathscr{I}$, $V \in \mathscr{C}$ and $h \in \mathrm{End}_{\mathscr{C}}(V \otimes X)$,

$$\mathrm{t}_{V \otimes X}(h) = \mathrm{t}_X(\mathrm{tr}_{\mathrm{L}}(h)).$$

Since $\mathscr{C}$ is ribbon, conditions 2R) and 2L) above are equivalent [GKP10].

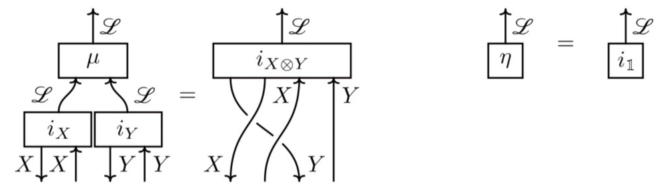
We say a trace t on an ideal $\mathscr{I} \subset \mathscr{C}$ is *non-degenerate* if for every $V \in \mathscr{I}$ and every $W \in \mathscr{C}$ the pairing $\mathrm{t}_V(\cdot \circ \cdot) : \mathscr{C}(W, V) \times \mathscr{C}(V, W) \to \Bbbk$ is non-degenerate. An important example of a tensor ideal is the projective ideal $\mathrm{Proj}(\mathscr{C})$. It is shown in Theorem 5.5 and Corollary 5.6 of [GKP18] that:

2.4. Coends and ends. We will now recall some well-known facts about the end of the functor $\mathscr{C} \times \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ sending every $(U, V) \in \mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ to $U \otimes V^* \in \mathscr{C}$ and about the coend of the functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ sending every $(U, V) \in \mathscr{C}$ to $U^* \otimes V \in \mathscr{C}$. We use the notation

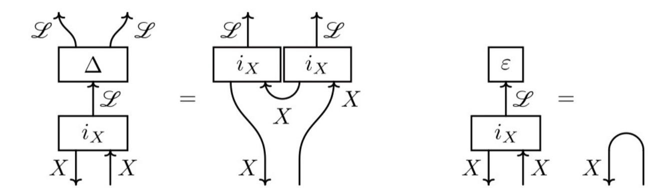
$$\mathscr{E} := \int_{X \in \mathscr{C}} X \otimes X^*, \qquad \mathscr{L} := \int^{X \in \mathscr{C}} X^* \otimes X,$$
$$j_X : \mathscr{E} \to X \otimes X^*, \qquad i_X : X^* \otimes X \to \mathscr{L},$$

for the end and the coend respectively, and for their corresponding dinatural transformations. See Sections IX.4–IX.6 of [M71] for a definition of dinatural

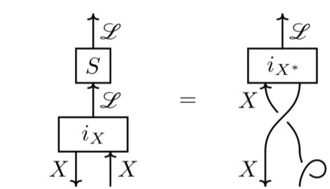
(3)



(4)



(5)



2.5. **Integrals and cointegrals.** Let us assume that $\mathscr{C}$ is in addition unimodular. A morphism $\Lambda \in \mathscr{C}(\mathbb{1}, \mathscr{L})$ is called a *right integral of* $\mathscr{L}$ if it satisfies

$$\mu \circ (\Lambda \otimes \mathrm{id}_{\mathscr{L}}) = \Lambda \circ \varepsilon. \tag{11}$$

A left integral of $\mathscr{L}$ is defined similarly[2]. It is known that right/left integrals of $\mathscr{L}$ exist and are unique up to scalar, see Proposition 4.2.4 of [KL01]. Furthermore,

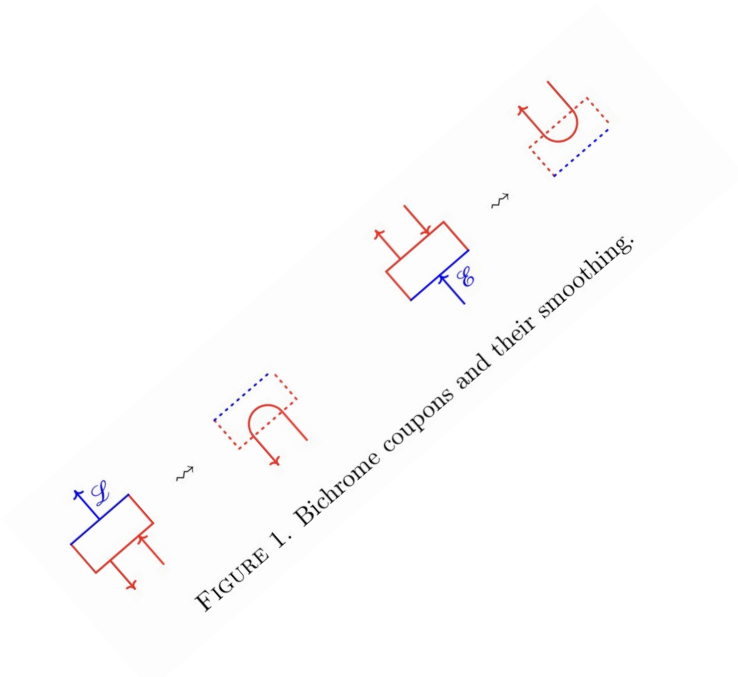


FIGURE 1. Bichrome coupons and their smoothing.

A 0-bottom graph is simply called a *bichrome graph*. See Figure 2 for an example of a 1-bottom graph together with its smoothing.

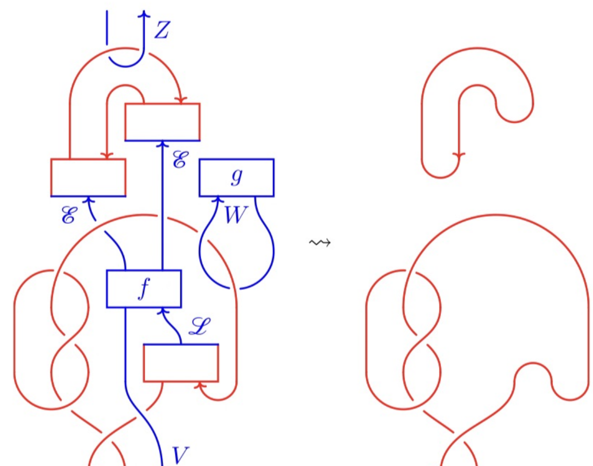


FIGURE 2. A 1-bottom graph and its smoothing.

$(X_1, Y_1, \ldots, X_n, Y_n) \in (\mathscr{C}^{\mathrm{op}} \times \mathscr{C})^{\times n}$. The n-dinatural transformation $\eta_{\tilde{T}}$ associates with every object $(X_1, \ldots, X_n) \in \mathscr{C}^{\times n}$ the morphism

$$F_{\mathscr{C}}(\tilde{T}_{(X_1,\ldots,X_n)}) \in \mathscr{C}(X_1^* \otimes X_1 \otimes \ldots \otimes X_n^* \otimes X_n \otimes F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V}), F_{\mathscr{C}}(\underline{\varepsilon}', \underline{V}')),$$

where $\tilde{T}_{(X_1,\ldots,X_n)}$ is the ribbon graph obtained from the n-bottom graph $\tilde{T}$ by labeling its kth cycle with X_k, by labeling every bichrome coupon intersecting it with either i_{X_k} or j_{X_k}, the structure morphisms of $\mathscr{L}$ and $\mathscr{E}$ defined in Section 2.4, for every integer $1 \leqslant k \leqslant n$, and by forgetting the distinction between red and blue. The universal property defining $\mathscr{L}$ implies the object $\mathscr{L}^{\otimes n} \otimes F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})$ equipped with the dinatural transformation $i^{\otimes n} \otimes \mathrm{id}_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})}$ is the coend for the functor $H_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})} \circ \sigma$. This determines a unique morphism $f_{\mathscr{C}}(\eta_{\tilde{T}}) \in \mathscr{C}(\mathscr{L}^{\otimes n} \otimes F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V}), F_{\mathscr{C}}(\underline{\varepsilon}', \underline{V}'))$ satisfying

$$f_{\mathscr{C}}(\eta_{\tilde{T}}) \circ (i_{X_1} \otimes \ldots \otimes i_{X_n} \otimes \mathrm{id}_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})}) = F_{\mathscr{C}}(\tilde{T}_{(X_1,\ldots,X_n)}). \tag{26}$$

Then we define $F_\Lambda(T) : F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V}) \to F_{\mathscr{C}}(\underline{\varepsilon}', \underline{V}')$ as

$$F_\Lambda(T) := f_{\mathscr{C}}(\eta_{\tilde{T}}) \circ (\Lambda^{\otimes n} \otimes \mathrm{id}_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})}). \tag{27}$$

Proposition 3.1. *$F_\Lambda : \mathscr{R}_\Lambda \to \mathscr{C}$ is a well-defined monoidal functor.*