Disclaimers:

• not contact expert
• no new facts
• four years ago
• last week

more info available at http://canyon23.net/math/
Outline:

1. TQFT framework
2. TQFT viewpoint for tight contact structures
3. Blob homology  (if there’s time)
1. TQFTs via topological fields and local relations
Key observation:

path integral $Z(B^{n+1}) \leftrightarrow$ local relations (skein-type relations)

- $Z(B^{n+1}; c) \in \mathbb{C}$, where $c \in C(\partial B^{n+1})$ is a boundary condition on $\partial B^{n+1}$
- $Z(B^{n+1}) : C(\partial B^{n+1}) \to \mathbb{C}$
- $Z(B^{n+1}) \in \mathcal{F}(\partial B^{n+1})$, where $\mathcal{F}(X^n)$ is the space of all functions from $C(X)$ to $\mathbb{C}$
- $Z(B^{n+1}) : \mathcal{F}(\partial_{\text{in}} B^{n+1}) \to \mathcal{F}(\partial_{\text{out}} B^{n+1})$
- changing notation, $\pi_D : \mathcal{F}(D) \to \mathcal{F}(D)$, where $D = B^n \approx \partial_{\text{in}} B^{n+1} \approx \partial_{\text{out}} B^{n+1}$
- $(D \times I) \cup_D (D \times I) \approx D \times I$ implies $\pi_D \circ \pi_D = \pi_D$
• Similarly, for any $n$-manifold $Y$, $Z(Y \times I)$ gives an operator $\pi_Y : \mathcal{F}(Y) \to \mathcal{F}(Y)$, and $\pi_Y \circ \pi_Y = \pi_Y$

• The Hilbert space for the theory is $Z(Y) \overset{\text{def}}{=} \text{im}(\pi_Y) \subset \mathcal{F}(Y)$

• Dually, define $A(Y) = \mathbb{C}[\mathcal{C}(Y)]/\sim$, where $x \sim 0$ if $f(x) = 0$ for all $f \in Z(Y)$
Let $D_i$ be an $n$-ball in $Y$. We can think of $\pi_{D_i}$ as an operator $\mathcal{F}(Y) \to \mathcal{F}(Y)$.

Since $Y \times I \cup D_i \times I \cong Y \times I$, we have $\text{im}(\pi_Y) \subseteq \text{im}(\pi_{D_i})$.

Since $Y \times I \cong D_0 \times I \cup \cdots \cup D_k \times I$, we have $\pi_Y = \pi_{D_0} \circ \cdots \circ \pi_{D_k}$ and therefore $\text{im}(\pi_Y) = \text{im}(\pi_{D_0}) \cap \cdots \cap \text{im}(\pi_{D_k})$.

It follows that

$$Z(Y) = \text{im}(\pi_Y) = \bigcap_{D \subseteq Y} \text{im}(\pi_D)$$
• Dually, $A(Y)$ is equal to finite linear combinations of fields modulo local relations

• Big conclusion: the path integral $Z(B^{n+1})$ is equivalent to local relations on fields on $B^n$. The path integral of any other $n+1$-manifold can be obtained by combinatorial methods.
Examples of fields and local relations

1. Fields: \( C(X) = \Xi \text{maps} X \to BG^3 \)

Local relations: \( f \sim g \) if \( f \) and \( g \) are homotopic (rel boundary)

\[ \Rightarrow \text{Dijkgraaf-Witten TQFT} \]
$n=2$

(2) Fields: $E(Y^2; c) = \Sigma 1$-submanifolds of $Y$ which restrict to $c$ on $dY^3$

e.g. $E(D^2; 6\text{pts})$ contains

Local relations: $0 \sim \sqrt{2} \cdot \phi$

\[ \begin{align*}
&\text{+ isotopy} \\
&\rightarrow \text{Turaev-Viro type TQFT}
\end{align*} \]
For each \((n-1)\)-manifold \(R^{n-1}\), we get a category \(A(R)\):

- **objects**: \(C(R)\) (fields on \(R\))
- **morphisms**: \(\text{mor}(a \to b) = A(R \times I; a, b)\)
- **composition**: gluing

\[ R \times I \]
2.2.2 More than one way to glue a collar
4.4.1 More gluing

Our goal is to describe $A(Y_{gl}; c)$ in terms of the various $A(Y; \cdot, \cdot, c)$ and the action of $A(-S \sqcup S) = A(S)^{op} \times A(S)$ on these spaces.

We'll start with the most abstract formulation of the codimension-1 gluing theorem, and then work our way toward more concrete statements.

**Theorem.** Let $Y$, $S$, $Y_{gl}$, $c$ be as above.

(a) For each object $x$ of $A(S)$ there is a map

$$gl_x : A(Y; \hat{x}, x, c) \to A(Y_{gl}; c).$$

(b) For each morphism $e : x \to y$ of $A(S)$ the following diagram commutes

$$\begin{array}{ccc}
A(Y; \hat{x}, x, c) & \xrightarrow{e \times 1} & A(Y; \hat{y}, x, c) \\
& & \\
& \downarrow{gl_x} & \\
A(Y_{gl}; c) & \xleftarrow{1 \times e} & A(Y_{gl}; c)
\end{array}$$
(c) \( A(Y_{gl}; c) \) is the universal object (vector space, set, or whatever flavor of \( A \)
we’re using) with properties (a) and (b). In other words, given a \( W \) and maps
\( gl'_x : A(Y; \hat{x}, x, c) \to W \) (for all \( x \)) such that the diagram analogous to the one in (b)
above commutes for all \( e \), there is a unique \( \theta : A(Y_{gl}; c) \to W \) such that \( gl'_x = \theta \circ gl_x \)
for all \( x \).

In other words, \( A(Y_{gl}; c) \) is the coend (see Appendix A [need more specific reference])
of the action of \( A(S)^{op} \times A(S) \) on \( A(Y; \cdot, \cdot, c) \).
Corollary. Let $Y$, $S$, $Y_{gl}$, $c$ be as above. If the target category of $A$ (on $n$-manifolds) is the category of vector spaces, then

$$A(Y_{gl}; c) = \left( \bigoplus_{x \in C(S)} A(Y; \hat{x}, x, c) \right) / \langle ev \sim ve \rangle.$$  

By $\langle ev \sim ve \rangle$ we mean the subspace of $\bigoplus_{x \in C(S)} A(Y; \hat{x}, x, c)$ generated by all $ev - ve$, for all morphisms $e : x \to y$ of $A(S)$ and all $v \in A(Y; y, x, c)$. Here we write the action of $A(S)$ as juxtaposition on the right and the action of $A(S)^{op}$ as juxtaposition on the left.

Note that $\bigoplus$ above means finite linear combinations.
$(n+1)$-dimensional part

What we want:

- $\mathbb{Z}(W^{n+1}); A(\partial W) \to \mathbb{C}$
- Can define inner product

$$\langle \cdot, \cdot \rangle : A(Y) \times A(Y) \to \mathbb{C}$$

$$\langle a, b \rangle \overset{\text{def}}{=} \mathbb{Z}(Y \times \Sigma)(\hat{a} \otimes b)$$

This $\mathbb{I}$. should be non-degenerate

- Gluing formula:

$$\mathbb{Z}(W_{ge})(b_{ge}) = \sum_i \mathbb{Z}(W)(b_i \hat{e}_i \otimes e_i) \cdot \frac{1}{\langle e_i, e_i \rangle}$$

($\{e_i\}$ = orthonormal basis of $A(N)$)
Thm. Choose \( z \in A(\mathbb{S}^n)^* \). If

1. induced I.P. on \( A(B^n; c) \) is positive definite \( \forall C \), and
2. \( \dim(A(\mathbb{S}^n; c)) < \infty \ \forall (\mathbb{S}, c) \)

then there is a unique partition function \( z \) such that
\( z(B^{n+1}) = z \in A(d \mathbb{B}^{n+1})^* \).

Proof:

\[ z \rightarrow \text{I.P. on } A(B^n; c) \rightarrow \text{gluing formula for 1-handles} \]
\[ \rightarrow \text{I.P. on } A(B^{n-1} \times S^1; c) \rightarrow \text{gluing formula for 2-handles} \]
\[ \rightarrow \text{I.P. on } A(B^{n-2} \times S^2; c) \rightarrow \text{gluing formula for 3-handles} \ldots. \]

Show independence under:

a) handle slides (easy)

b) handle cancellation (not hard)
<table>
<thead>
<tr>
<th>Fields</th>
<th>Local Relation</th>
<th>State Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maps into BG (G a finite group)</td>
<td>Homotopy of maps</td>
<td>Dijkgraaf-Witten sum on a triangulation</td>
</tr>
<tr>
<td>Pictures based on a disklike 2-category (e.g. a spherical category)</td>
<td>Isotopy plus relations coming from the category</td>
<td>Turaev-Viro sum</td>
</tr>
<tr>
<td>Pictures based on a ribbon category (a disklike 3-category)</td>
<td>Isotopy plus relations coming from the category</td>
<td>For a generic cell handle decomposition of a 4-manifold, the Crane-Yetter state sum</td>
</tr>
<tr>
<td>[same as above]</td>
<td>[same as above]</td>
<td>For 2-handles attached to the 4-ball, the Witten-Reshetikhin-Turaev surgery formula</td>
</tr>
<tr>
<td>[same as above]</td>
<td>[same as above]</td>
<td>For a “special spine” of a 4-manifold, the Turaev shadow state sum</td>
</tr>
</tbody>
</table>
The above techniques will work for a very general class of “topological fields” and local relations. We need only verify that a few simple axioms are satisfied, among them

- natural isomorphism $C(X \sqcup W) \cong C(X) \times C(W)$
- restriction maps $C(X) \to C(\partial X)$
- “product with $I$” maps $C(Y) \to C(Y \times I)$
- pushout diagram (fibered product)

```
\begin{tikzcd}
C(X) \arrow[dr] & C(Y) \arrow[dl] \\
C(X \sqcup_Y W) \arrow[urr] & C(W)
\end{tikzcd}
```

- local relations at least as strong as isotopy
2. TQFT viewpoint for tight contact structures
Def. A contact structure (C.S.) is a pair \((M^3, \xi)\) where:

- \(M^3 = \text{smooth oriented 3-manifold}\)
- \(\xi = \text{2-plane field \(CTM\)}\)
- \(\xi\) is \(\text{Ker} \alpha\), \(\alpha \in \Omega^1(M)\) (1-forms on \(M\))

and \(\alpha \land d\alpha > 0\) pointwise (independent of choice of \(\alpha\))

Standard CS on \(\mathbb{R}^3\): \(\alpha = dz + xdy\)
Thm. \( A \in M \) there exists a contact isomorphism \( f : \text{nbdl}(p) \rightarrow \mathbb{R}^3 \) taking \( p \) to \( 0 \in \mathbb{R}^3 \).

Let \( Y \subset M \). Then (generically) \( \xi |A_\eta \) gives rise to an oriented singular foliation of \( Y \), the characteristic foliation.
Thm. Let $Y_1 \subset M_1$, $Y_2 \subset M_2$, and suppose $\exists$ a diffeomorphism $f : Y_1 \rightarrow Y_2$ preserving characteristic foliations. Then $f$ can be extended to a contact isomorphism from a nbd of $Y_1$ in $M_1$ to a nbd of $Y_2$ in $M_2$. (CS in nbd of surface determined by characteristic foliation.)
Def: An **overtwisted disk** is a disk $D^2 \subset M$ (embedded, but not properly embedded) such that $\xi = TD$ along $\partial D$.

$M$ is **overtwisted** if it contains an O.T. disk.

$M$ is **tight** if it's not O.T.

Thm. The natural map

$$\{\text{O.T. CS on } M^3 \} \to \{\text{homotopy classes of plane fields on } M^3\}$$

is a homotopy equivalence.

$\Rightarrow$ Overtwisted CS's are well understood, tight CS's are the interesting case.
Key Observation: being tight/over-twisted is a local condition, so TQFT techniques can be applied.

BUT: We need to verify that contact structures satisfy the axioms for topological fields.
Def. A convex structure on $\mathcal{YCM}$ is a product nbd of $y$, $y \times I \in \text{nbhd}(y)$, such that the C.S. on $y \times I$ is invariant in the $I$ direction. $\mathcal{YCM}$ is convex if it has convex structure.

If $\mathcal{Y}$ has boundary then we require that the boundary be Legendrian with non-positive twisting.

Thm. If the Cof. of $\mathcal{Y}$ is Morse-Smale, then $\mathcal{Y}$ is convex.

Thm. Any $\mathcal{YCM}$ is $C^\infty$-close to a convex surface. ($\Rightarrow$ product-collar property of fields)
Category $A(Y)$:

objects = Morse-smale foliations on $Y$

morphisms = C.S. on $Y \times I$

What are the isomorphism classes of objects in $A(Y)$?

\[
\begin{align*}
fg &= \text{id}_a \\
gf &= \text{id}_b
\end{align*}
\]
Def. Let $YxI$ be a convex str. The dividing curve(s) of $YxI$ is a 1-submanifold of $Y$ for which the contact planes $\xi$ are vertical wrt. the $I$-direction.

Prop. Dividing curves are transverse to C.F and divide it into a maximal number of sink and source regions.

Thm. Let $a$ and $b$ be two characteristic foliations on $Y$. Then $a \bowtie b$ if $a$ and $b$ have isotopic dividing curves.
Theorem: Let $YCM$ be convex. Then a nbd of $Y$ is tight iff the dividing curves of $Y$ contain no trivial loops (or exactly one trivial loop if $Y \cong S^2$).

So objects in $A(Y)$ are, up to equivalence, the isotopy classes of oriented $1$-submanifolds of $Y$ (with $\pm$-shading) without excess trivial loops.
Corners

\[ \theta = \frac{D}{2} \]

\[ \theta = 0 \]
$A(D^2, \{2n \text{ pts} \})$

objects:

$n=1$

$n=2$

$n=3$

morphisms: Tight CS on $D^2 \times I = B^3$

(pinched):

Thm (Eliashberg): If a CS on a nbd of $\partial B^3$ is tight, then it has a unique extension to $B^3$
1 loop $\Rightarrow$ 
exists morphism (identity)

1 loop $\Rightarrow$ 
exists morphism (std elementary morphism)

3 loops $\Rightarrow$ 
only "zero" morphism
Drawing convention: shrink + region to 1-complex

N = 4
Definition of Elementary morphism:

Thm (Elshberg): Any morphism is the product of elementary morphisms.

Thm (Elshberg) Let \( q_0 \xrightarrow{e_1} q_1 \xrightarrow{e_2} q_2 \xrightarrow{\text{elementary}} \cdots \xrightarrow{e_i} q_i \)

Then \( e_1 \cdots e_i \) is tight iff each 2-sphere \( q_0, q_j \) \( 1 \leq j < i \) is tight.
• At this point we have an effective way of computing everything we want to know about the categories $A(D^2; 2n \text{ points})$.

• This means we can compute the results of gluing 3-manifolds along copies of $D^2$ in their boundary.

• In particular, we have an algorithm for computing all the tight contact structures on a handlebody (0- and 1-handles) which extend a given set of dividing curves on the boundary of the handlebody.

• (If we also understood the category $A(S^1 \times I; 2, 2)$, then we would be able to compute tight contact structures for any 3-manifold.)
(to the blackboard)