1. Review of Kh construction
2. Duality; 4-dimensional skein modules
3. Genus bounds for links
4. 0-tangles, 2-tangles and connected sums

Joint work with Scott Morrison

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1. The construction
Main idea: Follow Bar-Natan approach to Khovanov homology, but categorify quantum SU(2) skein relation instead of Kauffman skein relation.
Disoriented surfaces: piece-wise oriented, preferred normal direction on interface of oriented pieces, modulo fringe relations
Also impose neck cutting, sphere, torus relations:

\[ \frac{1}{2} \left( \begin{array}{c} \text{neck cutting} \\ \text{sphere} \\ \text{torus} \end{array} \right) = 0 \quad \text{and} \quad \begin{array}{c} \text{neck cutting} \\ \text{sphere} \end{array} = 2 \]
Coefficients

Any closed disoriented surface in $B^3$ is equivalent to some multiple of a disjoint union of zero or more genus three surfaces, so we’ll take our coefficient ring to be $\mathbb{C}[\alpha]$, where $\alpha$ is a genus three surface.

\[ \alpha = \]

In particular, the boundary connect sum of two punctured tori is $\frac{1}{2}\alpha$ times a disk (rather than zero). Note that $\alpha$ has degree $-4 = \chi(\text{genus 3 surface})$.

\[ = \frac{1}{2} \cdot \alpha \]

To recover the usual Khovanov homology, set $\alpha = 0$. For Lee homology, set $\alpha \neq 0, \alpha \in \mathbb{C}$. 
Let $c$ be a collection of oriented points in $\partial B^2$. Define a category $A_P(B^2; c)$ by

- **Objects**: Disoriented 1-manifolds in $B^2$ with boundary $c$.
- **Morphisms**: Disoriented surfaces in $B^2 \times I$, modulo the above relations.
- **Composition**: Gluing.

Define a category $\text{Mat}(A_P(B^2; c))$ by

- **Objects**: Tuples (formal direct sums) of objects of $A_P(B^2; c)$.
- **Morphisms**: Matrices of morphisms of $A_P(B^2; c)$.
- **Composition**: Matrix multiplication.

Define a category $\text{Kom}(A_P(B^2; c))$ by

- **Objects**: Chain complexes built out of $\text{Mat}(A_P(B^2; c))$.
- **Morphisms**: Chain maps modulo chain homotopy.
- **Composition**: (Obvious).
Fix $c \in \partial B^2 \subset \partial B^3$.

**Theorem.** There is a functor $Kh$ from the category of oriented tangles in $(B^3; c)$ and (isotopy classes of) isotopies between them to $\text{Kom}(A_P(B^2, c))$. Its graded Euler characteristic, appropriately interpreted, gives the Jones polynomial. It agrees with the original unoriented version of $Kh$, modulo the sign ambiguity for isotopies in that theory.

**Theorem.** The above functor extends to the category of oriented links in $B^3$ and oriented surface cobordisms (modulo isotopy) in $B^3 \times I$. 
Step 1: Define $K_h$ on generating objects

\[
\begin{align*}
\text{Diagram 1:} & \quad \rightarrow (q^{-2} \quad \rightarrow q^{-1}) \quad \rightarrow (\bullet) \\
\text{Diagram 2:} & \quad \rightarrow (q \quad \rightarrow q^2) \quad \rightarrow (\bullet)
\end{align*}
\]
Step 2: Define Kh on generating morphisms (Reidemeister moves)
Step 3: Show relations (movie moves) between generating morphisms are satisfied.
Want to extend definition to disoriented surface bordisms in $B^3 \times I$

New Reidemeister moves:
Need a morphism between a left fringe and a right fringe, but...

\[ \omega = \text{Diagram} = \omega^{-1} \]
Solution: Attach a spin framing to the morphism
Now we can define the chain maps
There are also more movie moves to check, for example

and
We can give an intrinsic geometric interpretation to the bigrading of Khovanov homology. One grading corresponds to Euler characteristic, and the other corresponds to framings of a tangle.

In terms of the traditional $t^r q^j$ bigrading, $f = 2r$ and $\chi = j - 3r$ (for knots).

If $\Sigma : T_1 \to T_2$ is a (framed) cobordism, then

$$K\text{h}(\Sigma) : K\text{h}(T_1) \to K\text{h}(T_2)$$

has $\chi$-degree equal to $\chi(\Sigma, T_1)$ (relative Euler characteristic) and $f$-degree given by the induced map between framings of $T_1$ and $T_2$. 
We have an exact triangle (long exact sequence)

\[
\begin{array}{c}
-1, 0 \\
\end{array}
\]

where the framing degrees are given relative to the blackboard framings of the tangles.
(This slide intentionally left blank)
2. Duality; 4-dimensional skein modules
For 2-categories, we have “Frobenius reciprocity”

\[ \text{mor}(a \cdot b \to c) \cong \text{mor}(a \to c \cdot \bar{b}) \]

These isomorphisms are coherent on the appropriate sense.
Khovanov homology satisfies a similar duality property:

\[ \text{mor}(\text{Kh}(P \bullet Q) \to \text{Kh}(R)) \cong \text{mor}(\text{Kh}(P) \to \text{Kh}(R \bullet \overline{Q})) \]
We'll begin with $Q = \underline{\kappa}$, a negative crossing oriented to the right. (The case for a positive crossing is exactly analogous.) Given a chain map $f \in \text{Hom}_{Kh}\left(\left\lbrack P \cdot \underline{\kappa}\right\rbrack, [R]\right)$, we'll produce the chain map

$$F(f) = (f \cdot 1_{\underline{\kappa}}) \circ (1_P \cdot R^2) \in \text{Hom}_{Kh}\left(\left\lbrack P\right\rbrack, \left\lbrack R \cdot \underline{\kappa}\right\rbrack\right).$$

We propose that the inverse of this construction is given by

$$\text{Hom}_{Kh}\left(\left\lbrack P\right\rbrack, \left\lbrack R \cdot \underline{\kappa}\right\rbrack\right) \ni g \mapsto F^{-1}(g) = (1_R \cdot R^{-1}) \circ (g \cdot 1_{\underline{\kappa}}).$$

The composition $F^{-1} \circ F$ applied to a chain map $f$ is

$$(1_R \cdot R^{-1}) \circ (((f \cdot 1_{\underline{\kappa}}) \circ (1_P \cdot R^2)) \cdot 1_{\underline{\kappa}}).$$
We also need to show that the above duality isomorphisms are coherent in the following sense. To a sequence of duality isomorphisms we can associate an isotopy between two links in $S^3$. If two different sequences of duality isomorphisms are associated to isotopic isotopies, then the compositions of the two sequences agree.
This is just functoriality for links in $S^3$. We’ve already proved functoriality for links in $B^3$, so what remains is

\[ \Rightarrow \]
We can now define an operadic product on $H_*(Kh(\cdot))$. (This is a 4-dimensional version of a planar algebra in the sense of Jones.) Let

$$W^4 = \mathcal{B}^4 \setminus \left( \bigsqcup_{1}^{n} \mathcal{B}^4 \right).$$

Let $\Sigma$ be a disoriented, framed surface properly embedded in $W$ with boundary

$$\partial \Sigma = L \cup \left( \bigsqcup_{1}^{n} \overline{L}_i \right).$$

Then we get a map

$$P_\Sigma : \bigotimes_{1}^{n} H_*(Kh(L_i)) \rightarrow H_*(Kh(L))$$

which satisfies the usual operad identities.
We can now define a “lasagna and meatballs” Stein module for a (spin) 4-manifold \( W \) with link \( L \subset \partial W \).

\[
\{ (B_i^4, L_i, x_i) \}
\]
\[
x_i \in H_* (K^n (L_i))
\]
\[
\Sigma \subset W \setminus \bigsqcup_i B_i^4
\]
\[
\partial \Sigma = L \cup L_1 \cup \cdots \cup L_n
\]

**Pictures/Relations**

1. isotopy
2. boring linearizing relations
3. \[ [\Sigma', \{ (B_j^4, L_i, x_i) \}] \rightarrow P_{\Sigma'} (\otimes x_j) \]
What does this have to do with a categorification of the Witten-Reshetikhin-Turaev TQFT?

- For general reasons we expect the Hilbert spaces of TQFTs to be dual to generalized skein modules (functions from generalized skeins to $\mathbb{C}$).

- Let $A_J(M^3)$ be the skein module based on the Jones/Kauffman skein relation. Let $A_{J,r}(M)$ be the finite quotient corresponding to an $r^{th}$ root of unity.

- A general procedure (which requires semisimplicity of $A_{j,r}$) constructs for each 4-manifold $W^4$ a function (path integral)

$$Z_{J,r}(W) : A_{J,r}(\partial W) \to \mathbb{C}$$

which satisfies TQFT gluing axioms. (i.e. we get a 3+1-dimensional TQFT.)
• Bad news: $A_{J,r}(M)$ is 1-dimensional if $M$ is closed.

• Good news: If $\partial W = M$, then $Z_{J,r}(W)(M, L)$ is equal to the Witten-Reshetikhin-Turaev invariant of $(M, L)$. (The choice of $W$ corresponds to a choice of framing of $M$.)

Our $A_{Kh}(W^3)$ is a categorification of $A_J(M^3)$, but we don’t have anything that corresponds to a choice of root of unity or finite quotient.
3. Genus bounds for links
4. 0-tangles, 2-tangles and connected sums
$A_{kh}(B^3; \phi)$

Objects: links in $B^3$
Morphisms: $\text{mor}(L_1 \to L_2) = H_*(kh(L_1; \mathbb{C}))$

\[
\begin{array}{cccccc}
\emptyset & \emptyset & \emptyset & \frac{1}{2} \emptyset & \emptyset & \emptyset \\
\text{objects: complexes based on formal direct sums of the empty link } \emptyset
\end{array}
\]

$\text{mor}(\emptyset \to \emptyset) = R = \mathbb{C}[\alpha] \rightarrow \text{Kom}(R)$
Any object in $\text{Kom}(R)$ is isomorphic to a direct sum of

$$E_{a,b} \quad O \rightarrow \emptyset \rightarrow O$$

(free)

$$C_{a^n, a, b} \quad O \rightarrow \emptyset \xrightarrow{\chi^n} \emptyset \xrightarrow{a, b} O$$

(torsion)
"Lee" theory

Change coefficients from \( \mathbb{C}[x] \) to \( \mathbb{C} \), \( x \) acts \( \neq 0 \).

\( \text{mor}(\text{a module}) \) becomes semisimple, 2-dim \& algebra, with idempotents

\[ e_{\pm} = \emptyset \pm \frac{1}{\sqrt{2n}} \top \]

\( \vdots \)

\( A_{\text{Lee}}(B^3; \emptyset) \)

\{ Objects: Links in \( B^3 \)

\{ Morphisms: oriented cobordisms in \( B^3 \times I \), modulo homology \}

\( H_*(\text{Kh}_{\text{Lee}}(L)) \cong \mathbb{C} \}

orientations of \( L^3 \) \( \cong \text{mor}_{\text{Lee}}(\emptyset \rightarrow L) \)
Keep framing-degree but lose $X$-degree
Suppose $\Sigma \subseteq B^4$, $d\Sigma = L$, framing $b$.

Then:

$E_{a,b} \in \text{K}h(L)$

$\otimes_{\text{C}(L)} \text{C} \rightarrow \text{K}h_{\text{Lee}}(L)$

$E_{0,0} \cong \text{K}h(\emptyset)$

$\otimes_{\text{C}(\emptyset)} \text{C} \rightarrow \text{K}h_{\text{Lee}}(\emptyset)$

$\chi$-degree of $\text{K}h(L)$ is $\chi(L)$, so

$\chi(L) \leq \varphi$ (Rasmussen bound)
(to the blackboard)