1. Deletion-contraction relations

2. Hard hexagons in the shadow world (and beyond)

3. Most topological state sums are instances of a more general construction
Deletion-Contraction

\[ G = \text{planar graph} \]
\[ I(G) = \text{graph invariant} \]
\[ I(G) \in \mathcal{C} \]

\[ A \cdot I(\mathcal{X}) + B \cdot I(\mathcal{Y}) + C \cdot I(\mathcal{Z}) = 0 \]

\[ A, B, C \in \mathcal{C}, \ A, B, C \neq 0 \]

Vertex+edge fugacities:
\[ I'(G) = I(G) \cdot \alpha^{\#e(G)} \cdot \beta^{\#v(G)} \]
\[ \rightarrow (A', B', C') = (A, \alpha B, \alpha \beta C) \]
In order to fully evaluate a graph, we further assume:

\[
- I(G_1 \cup G_2) = I(G_1) \cdot I(G_2)
- \phi = \lambda \cdot \phi
- * = \rho \cdot \phi
- \emptyset = \emptyset \cdot \phi
\]

Also,

\[
\frac{-B}{A} \cdot \phi - \frac{C}{A} = -\frac{B_\rho - C}{A}
\]

so for convenience, define

\[t = -\frac{B_\rho - C}{A}\]
Consider \( \{ \mathbf{N} = ) + \frac{A}{C} ) + \frac{B}{Ct} ; \)

\( \mathbf{N} \) is almost nilpotent.

\( \mathbf{N} = 0 \), \( \mathbf{N} = 0 \), but \( \mathbf{N} \neq 0 \)

So we impose further relation \( \mathbf{N} = 0 \Rightarrow \)

\( \mathbf{N} = L + \frac{A \lambda p}{C} + \frac{B p}{C} = 0 \)

(Can be thought of as a relation between \( L \) and \( \lambda \).)
Chromatic polynomial, \( \chi_k(G) = \# \text{ of vertex colorings with } k \text{ colors} \)

\[
\chi = \chi + \chi \implies A = 1, \quad B = -1, \quad C = 1
\]

\[
\chi = 0 \implies \lambda = 0
\]

\[
\cdot = k \implies \rho = k
\]

\[
\cdot = (k-1) \implies \tau = k
\]

\[
(0 = \ell = \frac{-A\lambda \rho}{C} - \frac{B\rho}{C} = k)
\]
Tutte polynomial (for connected graphs)

\[ \chi = \chi + \chi \Rightarrow A=1, B=-1, C=-1 \]

\[ \chi = y \Rightarrow \lambda = y \]

\[ \chi = x \Rightarrow \tau = x \Rightarrow \rho = 1-x \]

\[ 0 = \lambda = \frac{-A\lambda^2}{C} - \frac{B\rho}{C} = (y-1)(1-x) \]
Temperly-Lieb

- Choose \( u, v \in \mathcal{C}, u, v \neq 0 \)

- Choose \( \# \) such that \( \# = u^\ast \cdot (u + v) \cdot u \)

- Choose \( \beta \in \mathcal{C}, \beta \neq 0 \) (vertex fugacity)

- Define graph invariant by these substitutions:

\[ ) \rightarrow )) \]

\[ \begin{array}{c}
\begin{array}{c}
+ \\
- \\
\end{array}
\end{array} \rightarrow \beta \]

Special cases:

\[ \bullet \rightarrow \beta \cdot \circ = \beta \cdot 1 \]

\[ 1 \rightarrow \beta \cdot \circ \]
Then

\[ \beta, -\nu, -\beta \nu \rightarrow \beta, -\nu \beta^2, -\nu \beta^2 \]

\[ \Rightarrow A = 1, \ B = -\nu, \ C = -\beta \nu \]

\[ 0 \rightarrow \beta \cdot 0 = \beta d \quad \Rightarrow \quad \beta = \beta d \]

\[ \bigcirc \rightarrow d^2 \quad \Rightarrow \quad d = d^2 \]

\[ \rightarrow \quad \Rightarrow \quad \lambda = d\nu + v \]

\[ \text{(Check that } \lambda + \frac{A\lambda \rho}{C} + \frac{B\rho}{C} = 0) \]
Tutte polynomial via $T-L$:

$A = 1$  
$B = -V = -1$  
$C = -\beta d = -1$  
$\lambda = du + V = y$  
$\rho = \beta d = 1-x$  

$\Rightarrow$

$d^2 = (y-1)(1-x)$  
$\beta^2 = \frac{1-x}{y-1}$  
$u^2 = \frac{y-1}{1-x}$  
$V = 1$
Chromatic polynomial via T-L

\[ A = 1 \]
\[ B = -v = -1 \]
\[ C = -\beta u = 1 \]
\[ p = \beta d = \kappa \]
\[ \lambda = d^2 = \kappa \]

\[ d^2 = \kappa \]
\[ \beta = d \]
\[ u = -\frac{1}{d} \]
\[ v = 1 \]

\[ X_{d^2}(G) = \text{Yamada}(\hat{G}) \cdot d^{\#V(G)} \]

(recall:
\[ \psi = 0 \]
\[ \epsilon = \epsilon \])

(upto factors of $\beta = d$)
Hard Hexagons in the Shadow World

Want to evaluate "spin network":

\[ (\text{assume } G \text{ is trivalent}) \]

\[ \sum \left[ (\prod \Theta) \cdot (\prod \Theta)^{-1} \cdot (\prod O) \right] \]

\[ \sum \text{ restricted face labelings} \]

\[ = \sum \left[ \prod \Theta \cdot \prod \Theta^{-1} \cdot \prod O \right] \cdot \frac{1}{\sum O_i^2} \]

\[ 0 = \sum \delta_i^2 \]
If \( d = r = \frac{1 + \sqrt{5}}{2} \), then label set \( \{0, 2^3\} \).

\((0, 2, 0)\) is not an admissible triple, so...

\[ \Rightarrow \text{summation is over "hard" polygons.} \]
For general $d$ (but edge labels still all 2), label set $= \{0, 2, 4, 6, \ldots, 3\}$ and adjacent face labels differ by $-2, 0$ or $2$.

\[\text{Some sort of height model}\]
State sums via handle decompositions

- Turaev-Viro state sum
- Witten-Reshetikhin-Turaev state sum
- Crane-Yetter state sum (4-dim)
- Dijkgraaf-Witten state sum (any dimension)
- Turaev "shadow" state sum

Goal: Derive all off the above in a unified framework.

More specifically, show that all of the above arise from computing the path integral of a semi-simple TQFT in terms of a handle decomposition.
Ingredients for a TQFT:

- **top dimension** $n+1$
- system of "fields" (e.g. pictures) for manifolds of dimension $\leq n$
- **local relations** (at least as strong as isotopy) for fields on $n$-manifolds
Now define

\[ A(Y^n; c) = \mathbb{C}[\text{fields on } Y \text{ which restrict to } c \text{ on } \partial Y] / \langle \text{local relations} \rangle \]

(skein module)

Also define cylinder categories

\[ A(X^{n-1}) = \left\{ \begin{array}{l}
\text{objects} = \{ \text{fields on } X^3 \} \\
\text{morphisms} \quad a \rightarrow b = A(X \times I; \tilde{a}, \tilde{b}) \\
\text{composition} = \text{gluing}
\end{array} \right. \]

\[ A(\partial Y) \text{ acts on } \{ A(Y^n; c) \}_{c \in c} \text{ via gluing of collars} \]

\[ \rightarrow \text{ gluing formula for } n\text{-manifolds} \]

(for \( n=2 \), "particles" are irreps of \( A(S^1) \).)
$(n+1)$-dimensional part

what we want:

- \( \mathcal{Z}( W_+^\infty ); A(\partial W) \to C \)

- can define inner product

\[ \langle \cdot , \cdot \rangle : A(Y_+) \times A(Y) \to C \]

\[ \langle a, b \rangle \overset{\text{def}}{=} \mathcal{Z}( Y \times \Gamma ) (\hat{a} \circ \hat{b}) \]

this I. is should be non-degenerate

- gluing formula:

\[
\mathcal{Z}( W_{g_\varepsilon} ) ( b_{g_\varepsilon} ) = \sum_i \mathcal{Z}( W ) ( b \cdot \hat{e}_i \cdot v \cdot e_i ) \cdot \frac{1}{\langle e_i , e_i \rangle} \\
( \{ e_i \} = \text{orthogonal basis of } A(W) )
\]
Thm. Choose $z \in A(5^n)^*$. If

1. induced I.P. on $A(B^n; c)$ is positive definite $\forall c$, and
2. $\dim (A(4^n; c)) < \infty \quad \forall (4, c)$

then there is a unique partition function $z$ such that $z(B^{n+1}) = z \in A(dB^{n+1})^*$.

Proof:

$z \mapsto$ I.P. on $A(B^n; c) \mapsto$ gluing formula for 1-handles

$\mapsto$ I.P. on $A(B^{n-2} \times S^1; c) \mapsto$ gluing formula for 2-handles

$\mapsto$ I.P. on $A(B^{n-2} \times S^2; c) \mapsto$ gluing formula for 3-handles

show independence under:

a) handle slides (easy)
b) handle cancellation (not hard)
Example:

\[ n=2, \text{ fields = multi-curves, relations = d-isotopy} \]

(Temperly-Lieb)

- IP on \( B^2 \)

\[ \langle \circ, \circ \rangle = \frac{q}{b} = \theta_{abc} \]

- gluing a 1-handle

\[ \text{factor of } \frac{\theta_{abc}}{\theta_{ab} \cdot \theta_{bc}} = \theta^{-1} \]

- IP on \( S^1 \times I \)

\[ \langle \circ, \circ \rangle = \delta_{ab} \]
- gluing a 2-handle

$Z(M_{2h})(x)$

$= \sum_{q} Z(M)(xuq) \cdot d_{q}$

$\text{I.f. on } S^2$

$\langle \Phi_{S^2}, \Phi_{S^2} \rangle = Z(S^2 \times I) = Z(B^3 \cup 2h) = \sum_{q} Z(B^3) (O^q) \cdot d_{q} = \Xi d_{q}^2$

- gluing a 3-handle

$\text{factor of } \frac{1}{\langle \Phi_{S^2}, \Phi_{S^2} \rangle} = \frac{1}{\Xi d_{q}^2}$
Putting it all together...

If \( m^3 \) has a generic handle decomposition (dual to a triangulation), then

\[
\mathcal{Z}(M) = \sum \left[ \prod_{0-h} \bigotimes \prod_{2-h} \theta^{-1} \prod_{2h} O^\delta \prod_{3h} \left( \frac{1}{\epsilon d^2} \right) \right]
\]

labelings of 2-handles
<table>
<thead>
<tr>
<th><strong>Fields</strong></th>
<th><strong>Local Relation</strong></th>
<th><strong>State Sum</strong></th>
</tr>
</thead>
</table>
| Maps into $BG$ ($G$ a finite group)  
$n = \text{arbitrary}$ | Homotopy of maps | Dijkgraaf-Witten sum on a triangulation |
| Pictures based on a disklike 2-category (e.g. a spherical category)  
$n = 2$ | Isotopy plus relations coming from the category | Turaev-Viro sum |
| Pictures based on a ribbon category (a disklike 3-category)  
$n=3$ | Isotopy plus relations coming from the category | For a generic cell handle decomposition of a 4-manifold, the Crane-Yetter state sum |
| [same as above] | [same as above] | For 2-handles attached to the 4-ball, the Witten-Reshetikhin-Turaev surgery formula |
| [same as above] | [same as above] | For a “special spine” of a 4-manifold, the Turaev shadow state sum |