Configuration Spaces

of

Linkages

by

Kevin Walker

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Note: There is a gap in the proof of Prop. 3.3. See J.C. Hausmann, "Sur la topologie des bras articules".

§ 0 Preliminary Notions and Terminology.

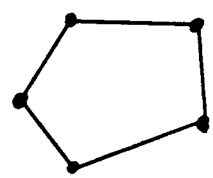
0.1 Graphs.

For our purposes, a *graph* G consists of a finite vertex set v(G) and an edge set $e(G) \subset v(G) \times v(G)$. Note that the edges are orientated. Write V(G) and E(G) for the cardinality of v(G) and e(G).

A *subgraph* of a graph G is a graph H such that $v(H) \subset v(G)$ and $e(H) \subset e(G)$.

The *degree* of a vertex $v \in v(G)$ is the number of edges incident to it (counted by multiplicity).

Let Σ_n denot the cyclic graph with n edges.



0.2 Fibered Products.

(Here and throughout, all maps are continuous maps of topological spaces.)

Given maps $f_i : A_i \to B$, $1 \leqslant i \leqslant n$, the *fibered product* of $\{f_i\}$ is a space Z together with maps $p_i : Z \to A_i$ such that

1) each square

$$\begin{tikzcd} & Z \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ A_i \arrow[dr, "f_i"'] & & A_j \arrow[dl, "f_j"] \\ & B & \end{tikzcd}$$

commutes.

2) given H and $g_i : H \to A_i$ such that $g_i f_i = g_j f_j$ for all i, j, there is a unique map $\lambda : H \to Z$ such that $g_i = p_i \lambda$ for all i.

More concretely, $Z = \{(x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n : f_1(x_1) = \ldots = f_n(x_n)\}$ and p_i is the restriction of the projection. (See [Spanier, p. 98].)

Example. Let A and B be finite sets. The $\mathbf{R}^{A \cup B}$ is the fibered product of $\mathbf{R}^A$ and $\mathbf{R}^B$ over $\mathbf{R}^{A \cap B}$, with respect to the restriction maps.

0.3 Cohomology Transfer.

Let M, N, be compact oriented manifolds without boundary and $D_M : H_*(M) \cong H^*(M)$, $D_N : H_*(N) \cong H^*(N)$ be the Poincare duality ismorphisms. Given $f : M \to N$, define the *cohomology transfer homomorphism* $f^! : H^*(m) \to H^*(N)$ to be $D_N f_* D_M^{-1}$.

Let M, N, L be compact oriented manifolds without boundary, $f : M \to N$, $g : N \to L$. Then

$$(fg)^! = \cancel{g^! f^!}\ f^! g^! \qquad 0.3.1$$

If $x \in H^*(N)$, $y \in H^*(M)$, then

$$f^!(f^*(x) \cup y) = x \cup f^!(y) \qquad 0.3.2$$

Let $K \subset N$ be closed, $f(M) \subset K$, $i : K \to N$ the inclusion. Then

$$f^! = i^* f^! \qquad (: H^*(M) \to H^*(K)) \qquad 0.3.3$$

(Here $f^! : H^*(M) \to H^*(K)$ is the composition

$$H^*(M) \xrightarrow{D_m^{-1}} H_*(M) \xrightarrow{f_*} H_*(N, N-K) \xrightarrow{D} H^*(K))$$

(See [Dold, VIII.10].)

§ 1 Definitions.

Definition. A *linkage* is a pair (G,l) consisting of a graph G and a function l: $e(G) \to \mathbf{R}_+ = \{x \in \mathbf{R}: x \geqslant 0\}$.

If $L = (G,l)$, we sometimes write $e(L)$ for $e(G)$ and $v(L)$ for $v(G)$.

Definition. Given a linkage $L = (G,l)$, the *free configuration space* of L, denoted F(L), is

$$\{p \in (\mathbf{R}^2)^{v(G)} : |p_v - p_w| = l((v,w)) \text{ for all } (v,w) \in e(G)\},$$

with the induced topology.

Thinking of L as a collection of bars and pivots, F(L) is the space of all positions of L in the plane.

Examples. The free configuratiuon spaces of the linkages in figure 1 are, as the indicated parameters should make clear,

(a) $\mathbf{R}^2$
(b) $\mathbf{R}^2 \times S^1$
(c) $\mathbf{R}^2 \times S^1 \bigcup \mathbf{R}^2 \times S^1$ (disjoint union)
(d) $\mathbf{R}^2 \times (S^1)^4$

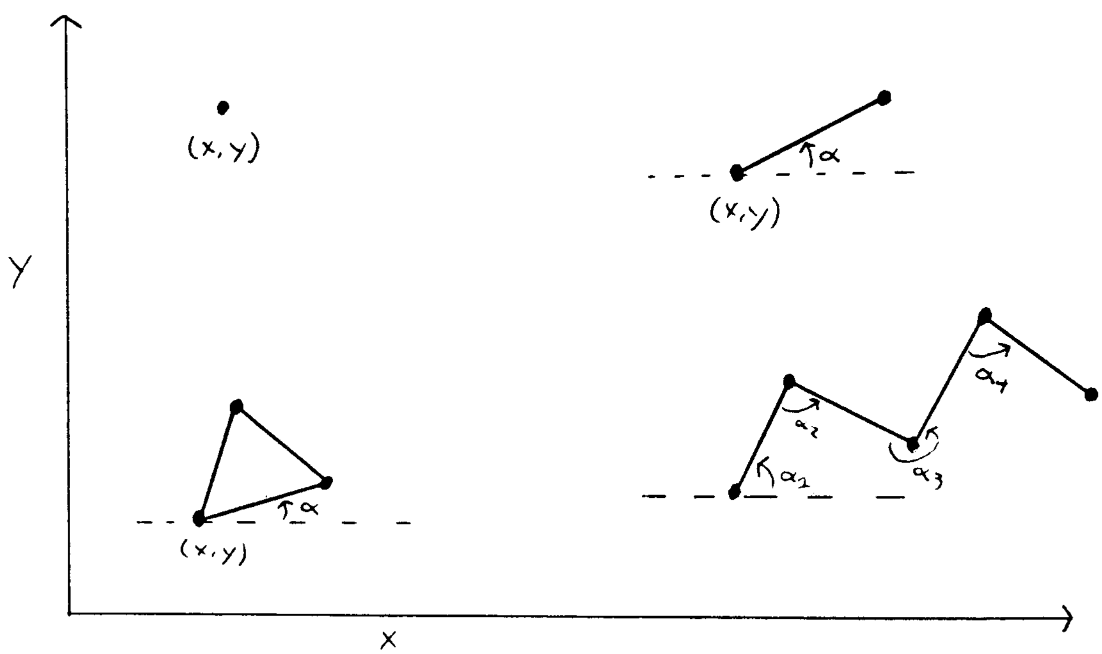


Figure 1

Let $\Gamma = \mathbf{R}^2 \times S^1$ be the group of orientation preserving isometries of $\mathbf{R}^2$. Γ acts on $(\mathbf{R}^2)^{v(G)}$ component-wise. If $p \in F(L) \subset (\mathbf{R}^2)^{v(G)}$ and $\gamma \in \Gamma$, then $\gamma(p) \in F(L)$ also (Γ preserves lengths). However, this is not a significantly different position of L, since it corresponds to holding L rigid while translating and rotating it. This motivates

Definition. The *configuration space* of a linkage L, denoted $\mathcal{C}(L)$, is

$$F(L) / \Gamma,$$

where Γ acts on $F(L)$ as described above.

If L contains a bar of non-zero length, "nailing down" the bar determines a representative of $c \in \mathcal{C}(L)$ in $F(L)$. Thus $\mathcal{C}(L)$ may be thought of as the space of positions of of L in $\mathbf{R}^2$ with one bar fixed.

Examples. The configuration spaces of the linkages in figure 2 are

(a) point
(b) $\mathbf{R}_+$
(c) point
(e) $(S^1)^4$
(f) $(S^1)^2$
(g) 2 points = S^0

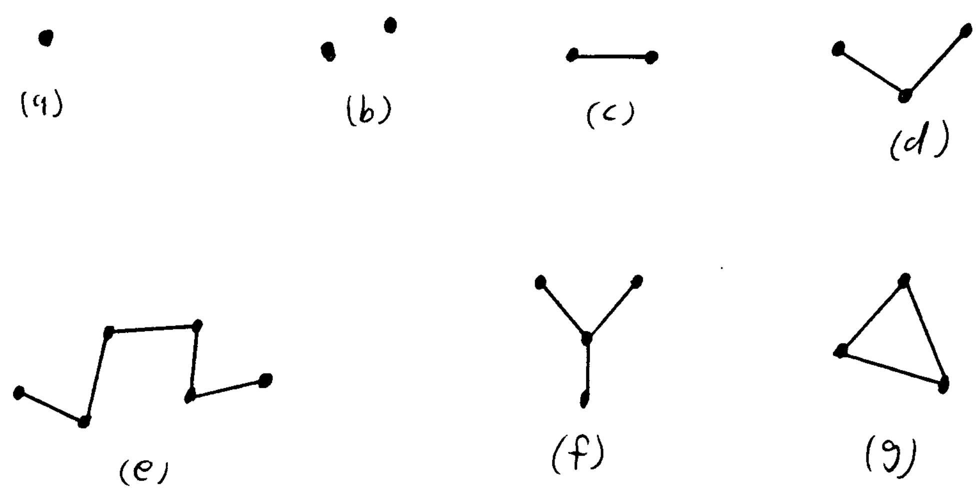
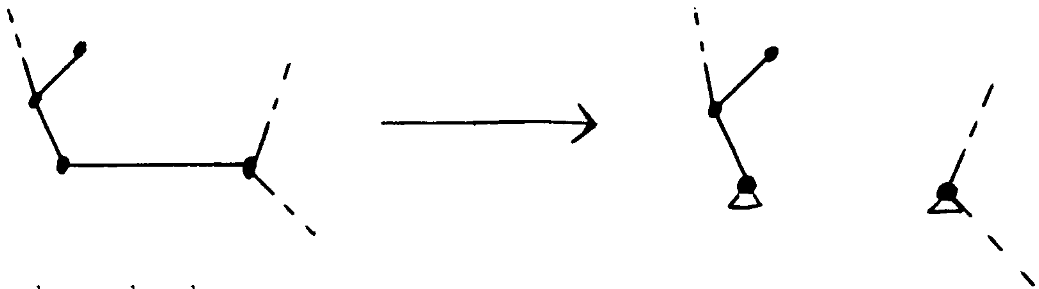


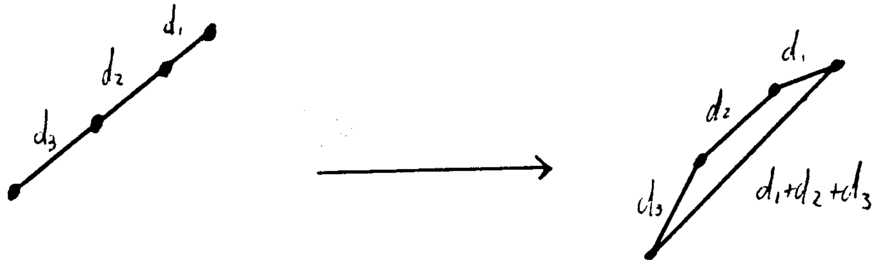
Figure 2.

(d) S^1

Remark. The usual definition of a linkage involves anchor points and pivots in the middle of bars, as well as no group action. The (components of) the configuration spaces obtained via the two definitions are the same, however (as long as there is a non-zero bar). Schematically, to go from a linkage-as-defined-here to a traditional linkage,



and to go the other way



We want to study how the topology of $C(G,l)$ changes as we change the length function l. Since multiplying l by a positive constant has no effect on the topology of $C(G,l)$, we consider l modulo this equivalence relation. Equivalently, we can require

$$\sum_{e\in e(G)} l(e) = 1.$$

(We ignore the case $l(e) = 0$ for all $e \in e(G)$.) The set of all such functions l is just Δ^{E-1}, the E-1 simplex, where $E = E(G)$. We can think of Δ^{E-1} as a base space, with fiber $C(G,l)$ lying above each $l \in \Delta^{E-1}$.

More precisely, let $v(G) = \{1, 2, \ldots V\}$ and $e(G) = \{(v_1,w_1), \ldots (v_E,w_E)\}$, and define

$$\tilde{p}_G : (\mathbf{R}^2)^{v(G)} \to \mathbf{R}_+^{e(G)} = \{x \in \mathbf{R}^{e(G)} : x_e \geqslant 0 \text{ for all } e \in e(G)\}$$

$$(u_1, \ldots, u_V) \to (|u_{v_1} - u_{w_1}|, \ldots, |u_{v_E} - u_{w_E}|).$$

If $l \in \mathbf{R}_+^{e(G)}$, $\tilde{p}_G^{-1}(l)$ is $F(G,l)$.

Since $\tilde{p}_G$ is invariant under the action of Γ on $(\mathbf{R}^2)^{v(G)}$, we get a map

$$\hat{p}_G : (\mathbf{R}^2)^{v(G)}/\,\Gamma \to \mathbf{R}_+^{e(G)},$$

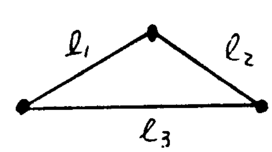
and $\hat{p}_G^{-1}(l) = C(G,l)$.

Let $\mathbf{R}_+^\times$ be the multiplicative group of positive reals. $\mathbf{R}_+^\times$ acts on $(\mathbf{R}^2)^{v(G)}/\,\Gamma$ and $\mathbf{R}_+^{e(G)}$ in the obvious way, and the actions commute with $\hat{p}_G$. Therefore we get a map

$$p_G : ((\mathbf{R}^2)^{v(G)}/\,\Gamma)/\,\mathbf{R}_+^\times \to \mathbf{R}_+^{e(G)}/\,\mathbf{R}_+^\times$$

induced by $\hat{p}_G$. But $\mathbf{R}_+^{e(G)}/\,\mathbf{R}_+^\times$ is $\Delta^{e(G)-1}$, and $((\mathbf{R}^2)^{v(G)}/\,\Gamma)/\,\mathbf{R}_+^\times = (\mathbf{C}^{v(G)}/\,\mathbf{R}^2)/\,(S^1 \times \mathbf{R}_+^\times) = \mathbf{C}^{v(G)-1}/\,\mathbf{C}^\times = \mathbf{CP}^{v(G)-2}$, so we actually have a map

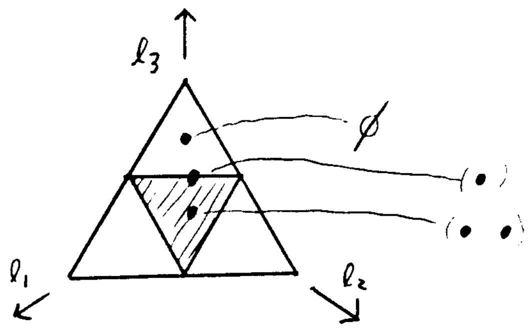
$$p_G : \mathbf{CP}^{v(G)-2} \to \Delta^{e(G)-1}.$$



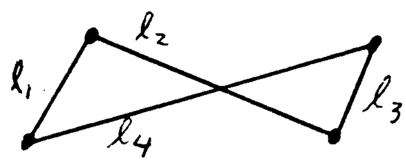
Examples. (a) The triangle (Σ_3). let l_1, l_2 and l_3 be the lengths of the sides, and $l_1+l_2+l_3 = 1$. It is easy to see that $C(\Sigma_3,l)$ is

(i) $\varnothing$, if l_1, l_2 or $l_3 > 1/2$
(ii) 1 point, if l_1, l_2 or $l_3 = 1/2$
(iii) 2 points, if $l_1, l_2, l_3 < 1/2$.

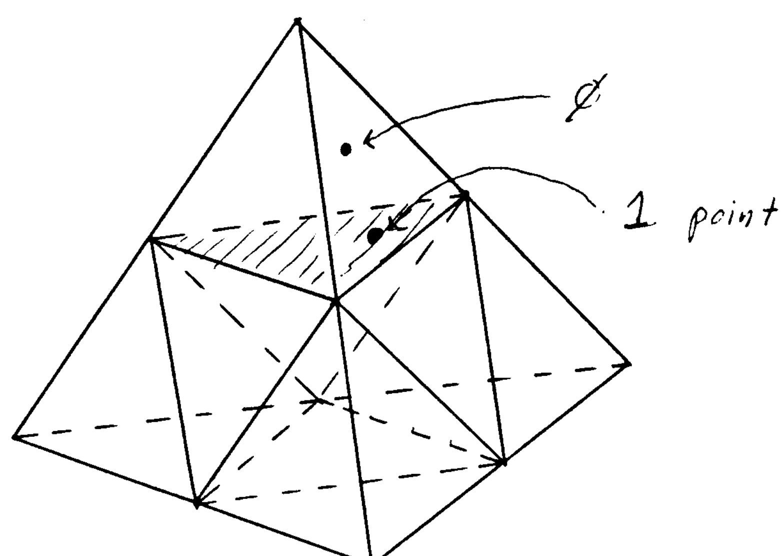
Represent $\mathbf{CP}^1 = S^2$ as 2 equilateral triangles glued together via the identity map on the boundary. Then $p_{\Sigma_3} : \mathbf{CP}^1 \to \Delta^2$ maps these triangles onto the central triangle $l_1, l_2, l_3 \leqslant 1/2$ of Δ^2:

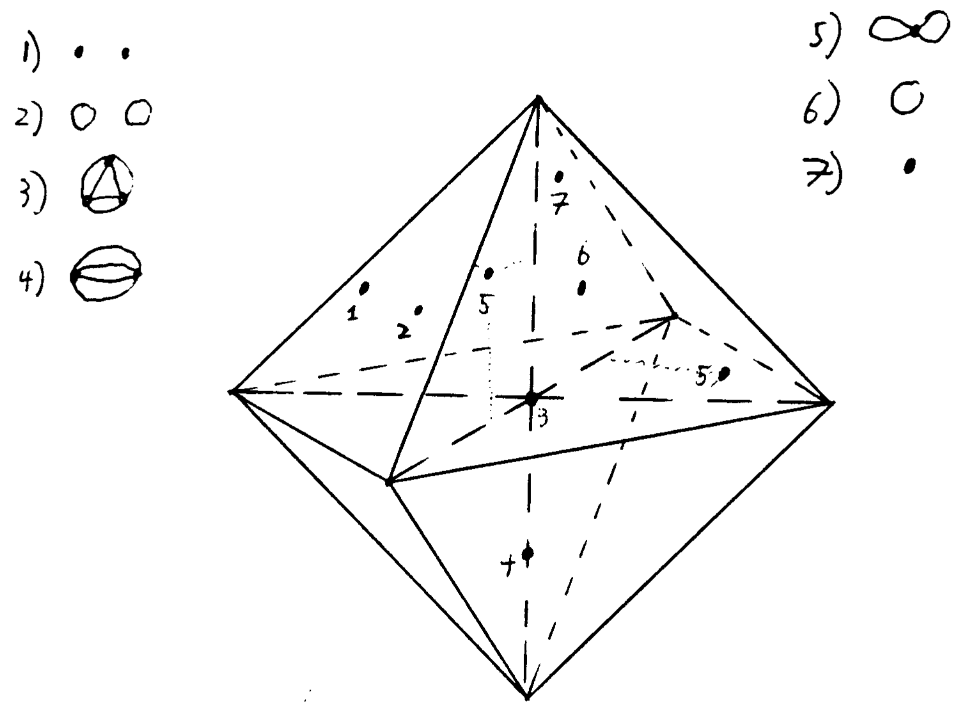


(b) The quadrilateral Σ_4.



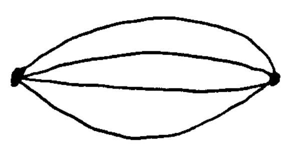
Δ^3 is a tetrahedron. The triangles $l_i = 1/2$, $1 \leqslant i \leqslant 4$, truncate the corners of Δ^3. On these triangles, $C(\Sigma_4, l)$ = 1 point, and in the corners, $C(\Sigma_4, 1) = \emptyset$. The squares $l_1+l_2=l_3+l_4$, $l_1+l_3=l_2+l_4$, $l_1+l_4=l_2+l_3$ divide the remaining octahedron into four octants. In the interiors of the four octants where the sum of the greatest side and the least side is greater than the the sum of the other two sides, $C(\Sigma_4, 1) = S^1$. In the interiors of the other four octants,



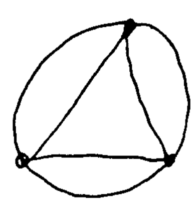


$C(\Sigma_4, 1)$ = 2 disjoint S^1 's. If l lies in only one square, $C(\Sigma_4, 1)$ is two circles with one point identified.

If l is in exactly two squares, $C(\Sigma_4, 1)$ is two circles with two points identified.



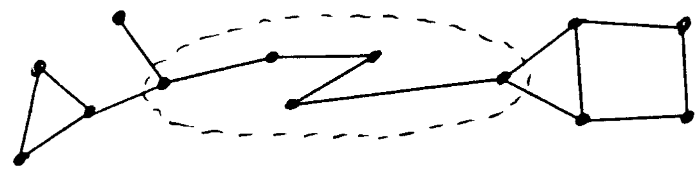
At the center (the intersections of all three squares), $C(\Sigma_4, 1)$ is two circles with three points identified.



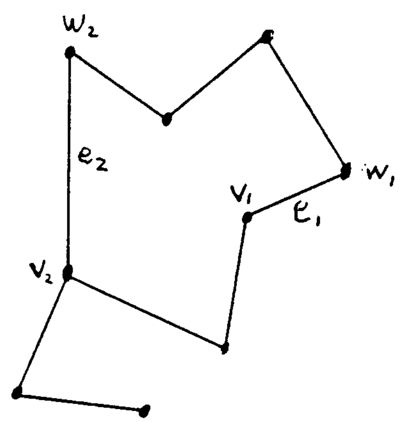
On the interiors of the faces of the octahedron where $l_i = 0$, $1 \leqslant i \leqslant 4$, $C(\Sigma_4, 1) = 2$ points.

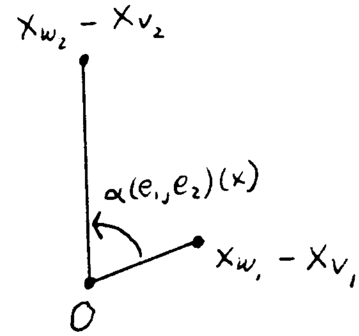
§ 2. Basic results.

Definition. A *chain* in a graph is a sequence of edges $(v_0,v_1), (v_1,v_2), \ldots, (v_{n-1},v_n)$ such that the degree of $v_i = 2$, $1 \leqslant i \leqslant n-1$.



Let $e_1 = (v_1,w_1)$, $e_2 = (v_2,w_2) \in e(G)$. Define $\alpha(e_1,e_2) : F(G,l) \to S^1$ by $\alpha(e_1,e_2)(x) =$ angle from $x_{w_1} - x_{v_1}$ to $x_{w_2} - x_{v_2}$ in $\mathbf{R}^2$.





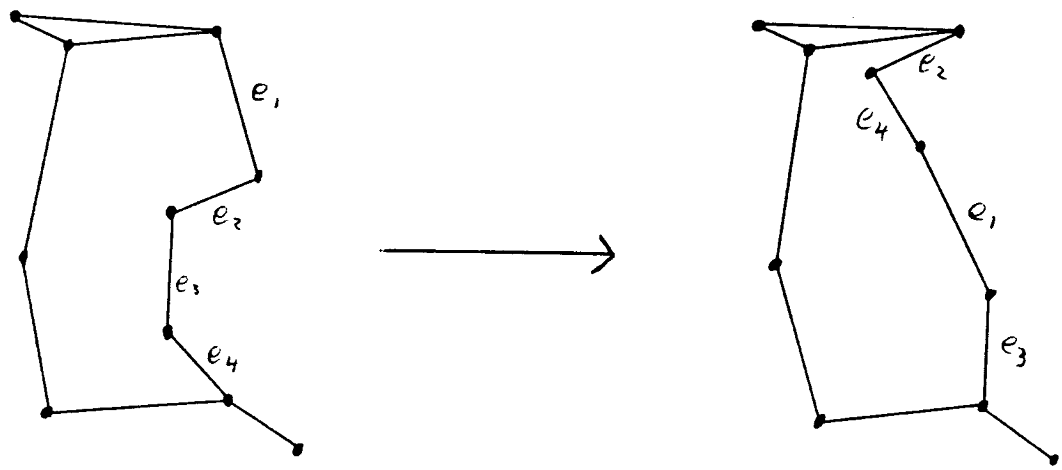
Since $\alpha(e_1,e_2)$ is invariant under the action of Γ on $F(G,l)$, it induces a map $C(G,l) \to S^1$, also denoted $\alpha(e_1,e_2)$.

Theorem 2.1. Let $e_1, e_2, \ldots, e_n$ be a chain in G and let π be a permutation of $1, \ldots, n$. Let $l : e(G) \to \mathbf{R}_+$ be a length function of G and let

$$l'(f) = \begin{cases} l(e_{\pi_i}), & \text{if } f = e_i \text{ for some } i \\ l(f), & \text{otherwise} \end{cases}$$

Then $F(G,l) \cong F(G,l')$ and $C(G,l) \cong C(G,l')$. Furthermore, there is a unique homeomorphism $C(G,l) \to C(G,l')$ which preserves the angle functions $\alpha(g,h)$, $g, h \in e(G)$.

Proof. An informal argument will be more enlightening than a rigorous proof. Think of the edges of the chain as free vectors in the plane. There is a unique way to arrange them in the new order without changing their length or direction. Since addition in $\mathbf{R}^2$ is commutative, the ends of the chain stay fixed, and all the connections in the linkage are preserved. $\boxtimes$

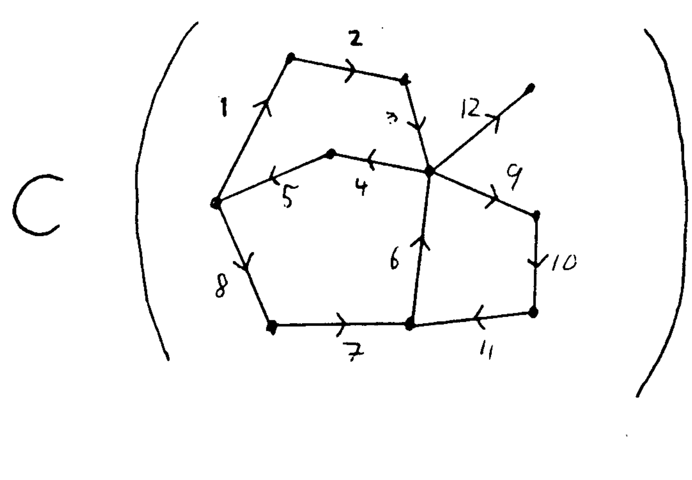


Corollary 2.2. The configuration space of a cyclic linkage (i.e. a linkage whose graph is cyclic) depends only on the set of lengths of the sides, not on their order. $\boxtimes$

Remark. The above suggests another way of looking at configuration spaces of linkages. We can think of the bars as vectors in $\mathbf{R}^2$ of fixed length. Each cycle in the underlying graph of the linkage constrains the vectors (= edges) in that cycle to add up to zero. For example, $C(\Sigma_n, l)$ is

$$\{(\theta_1, \ldots, \theta_n) \in (S^1)^n : \sum_{j=1}^{n} l_j e^{2\pi i\theta_j} = 0, \theta_1 = 0\}$$

(We need the $\theta_1 = 0$ constraint to get rid of unwanted global rotations.) Similarly,



is

$$\{(\theta_i) \in (S^1)^{12} : \sum_{j=1,2,3,4,5} l_j e^{2\pi i \theta_j} = 0, \sum_{j=4,5,8,7,6} l_j e^{2\pi i \theta_j} = 0, \sum_{j=6,9,10,11} l_j e^{2\pi i \theta_j} = 0, \theta_1 = 0\}$$

In general, any configuration space $C(G,l)$ is the intersection of the $E(G)$-torus

$$\{(z_i) \in (\mathbf{R}^2)^{E(G)} : |z_i| = l_i \text{ forall } i\} \subset (\mathbf{R}^2)^{E(G)}$$

with an affine subset. This affine subset is the intersection of codimension 2 hyperplanes; one for each (independant) cycle of G and one more to get rid of global rotations.

Definition. If $L=(G,l)$ is a linkage and H is a subgraph of G, then $K=(H, l|_{e(H)})$ is a *sublinkage* of L. Write $K \subset L$.

If $K \subset L$, the natural projection $(\mathbf{R}^2)^{v(L)} \to (\mathbf{R}^2)^{v(K)}$ induces *restriction maps*

$$\tilde{\pi}_K^L : F(L) \to F(K)$$

$$\pi_K^L : C(L) \to C(K).$$

If $J \subset K \subset L$, it is clear that $\tilde{\pi}_J^L = \tilde{\pi}_J^K \tilde{\pi}_K^L$ and $\pi_J^L = \pi_J^K \pi_K^L$.

The following theorems are the basic tools for determining configuration spaces of linkages

Theorem 2.3. Given linkages $J, K_1, K_2, \ldots, K_n \subset L$ such that $K_i \bigcap K_j = J$ for all i, j, and $K_1 \bigcup \ldots \bigcup K_n = L$, $F(L)$ together with the maps $\{\tilde{\pi}_{K_i}^L ; F(L) \to F(K_i)\}$ is the fibered product of C $\{\tilde{\pi}_J^{K_i} : F(K_i) \to F(J)\}$.

Proof. Let H be a space and $\{g_i : H \to F(K_i)\}$ be maps satisfying $g_i \tilde{\pi}_J^{K_i} = g_j \tilde{\pi}_J^{K_j}$ for all i, j. We must must show that there is a unique map $\lambda : H \to F(L)$ such that $g_i = \tilde{\pi}_{K_i}^L \lambda$ for all i.

Consider the following diagram:

$$\begin{tikzcd}
 & F(L) \arrow[r, hook] \arrow[d, "\pi^L_{K_i}"'] & (\mathbb{R}^2)^{v(L)} \arrow[d, "p_i"] \\
H \arrow[r] \arrow[urr, dashed, "\lambda"] & F(K_i) \arrow[r, hook, "i_{K_i}"'] \arrow[d, "\pi^{K_i}_J"'] & (\mathbb{R}^2)^{v(K_i)} \arrow[d] \\
 & F(J) \arrow[r, hook] & (\mathbb{R}^2)^{v(J)}
\end{tikzcd}$$

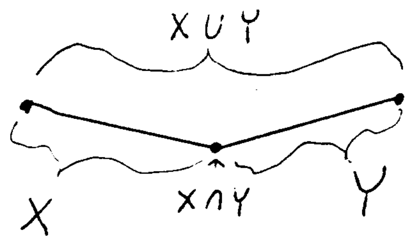
Since $(\mathbf{R}^2)^{v(L)}$ is the fibered product of $\{(\mathbf{R}^2)^{v(K_i)}\}$ with respect to the natural projections, there is a unique map $\lambda : H \to (\mathbf{R}^2)^{v(L)}$ such that $i_{K_i} g_i = p_i \lambda$ for all i.

It suffices to show that $\lambda(H) \subset F(L)$. Let $c = \lambda(h)$, $h \in H$. Let $e = (v,w) \in e(L)$. Since $L = \bigcup K_i$, $e \in e(K_i)$ for some i. Then

$$
\begin{aligned}
|x_v(c) - x_w(c)| &= |x_v(p_i(c)) - x_w(p_i(c))| \\
&= l(e)
\end{aligned}
$$

since $p_i(c) = g_i(h) \in F(K_i)$. Since e was arbitrary, $c \in F(L)$. ⊠

The naive generalization of theorem 2.3 to configuration spaces – replacing the F's by C's and the $\tilde{\pi}$'s by π's – does not work. Consider, for example, the linkage



$C(X) = C(Y) = C(X \bigcap Y) =$ point, but $C(X \bigcup Y) = S^1$. The problem is that the stabalizer of $F(X \bigcap Y)$ in Γ is not the identity, and hence elements $x \in C(X)$, $y \in C(Y)$ and $z \in C(X \bigcap Y)$ such that $\pi^X_{X \cap Y}(x) = z = \pi^Y_{X \cap Y}(y)$ do not determine a unique point in $C(X \bigcup Y)$.

Let J, $K_1, \ldots, K_n$ and L be as in theorem 2.3. Let

$$F^*(J) = \{x \in F(J) : \mathrm{stab}(x) = 1 \in \Gamma\}$$

$$F^*(K_i) = (\tilde{\pi}_J^{K_i})^{-1}(F^*(J))$$

$$F^*(L) = (\tilde{\pi}_J^L)^{-1}(F^*(J))$$

Let

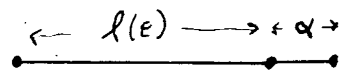
$$C^*(X) = F^*(X)/\,\Gamma \subset C(X),\ X = J, K_1, \ldots, K_n, L$$

Theorem 2.4. $C^*(L)$ with $\{\pi^L_{K_i}|_{C^*(L)} : C^*(L) \to C^*(K_i)\}$ is the fibered product of $\{\pi_J^{K_i}|_{C^*(K_i)} : C^*(K_i) \to C^*(J)\}$.

Proof. Given $z \in C^*(J)$, $y_i \in C^*(K_i)$, $1 \leq i \leq n$, such that $\pi_J^{K_i}(y_i) = z$ for all i, we must show that there is a unique $x \in C^*(L)$ such that $\pi_{K_i}^L(x) = y_i$ for all i. Clearly such elements exist. Let x and x' be two such.

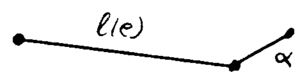
We can choose representatives $\tilde{x}, \tilde{x}' \in F^*(L)$ such that $\tilde{\pi}_J^L(\tilde{x}) = \tilde{\pi}_J^L(\tilde{x}')$. Suppose $\tilde{x} \neq \tilde{x}'$. Then, by 2.3, there is an i such that $\tilde{\pi}_{K_i}^L(\tilde{x}) \neq \tilde{\pi}_{K_i}^L(\tilde{x}')$. But $[\tilde{\pi}_{K_i}^L(\tilde{x})] = \pi_{K_i}^L(x) = y_i = \pi_{K_i}^L(x') = [\tilde{\pi}_{K_i}^L(\tilde{x}')]$. So there is a $\gamma \in \Gamma$ such that $\gamma\tilde{\pi}_{K_i}^L(\tilde{x}) = \tilde{\pi}_{K_i}^L(\tilde{x}')$. This implies $\gamma\tilde{\pi}_J^L(\tilde{x}) = \tilde{\pi}_J^L(\tilde{x}')$. Since $\tilde{\pi}_J^L(\tilde{x}) = \tilde{\pi}_J^L(\tilde{x}')$ and $\mathrm{stab}(\tilde{z}) = 1$ for all $\tilde{z} \in F^*(J)$, $\gamma = 1$. So $\tilde{\pi}_{K_i}^L(\tilde{x}) = \tilde{\pi}_{K_i}^L(\tilde{x}')$ after all. Therefore $\tilde{x} = \tilde{x}'$ and $x = [\tilde{x}] = [\tilde{x}'] = x'$. ☒

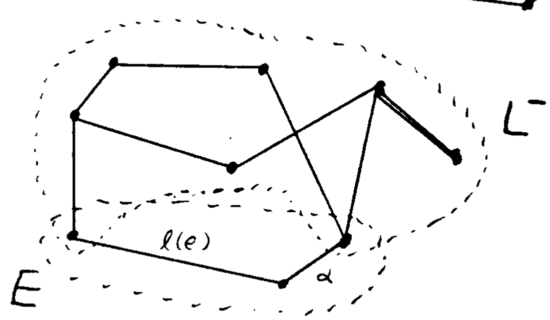
Corollary 2.5. Let $L = (G,l)$. Let $e \in e(G)$ and $l_\epsilon(f) = l(f) + \epsilon$ if $f = e$ and $l(f)$ otherwise. If $C(G,l_\epsilon) = C(G,l)$ for $-\delta \leq \epsilon \leq \delta$, then $C(L') = C(L) \times S^1$, where L' is obtained from L by replacing e by



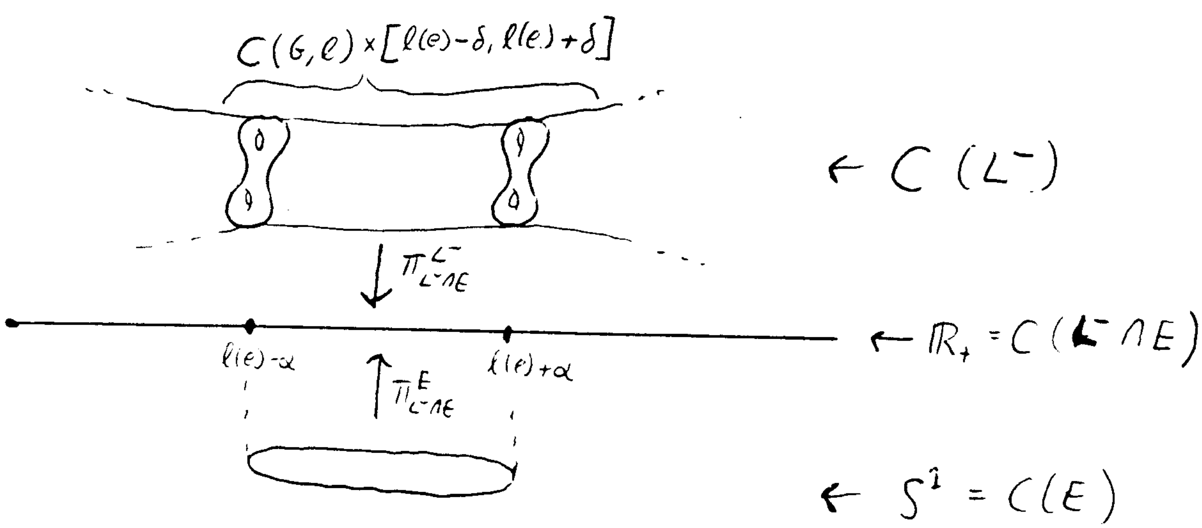
$0 < \alpha < \delta$.

Proof. Let $L^- = L - e$ and E =



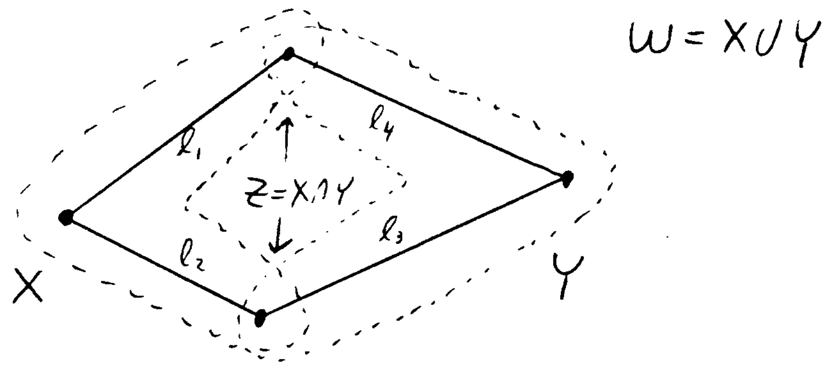


Then the corresponding fibered product looks like



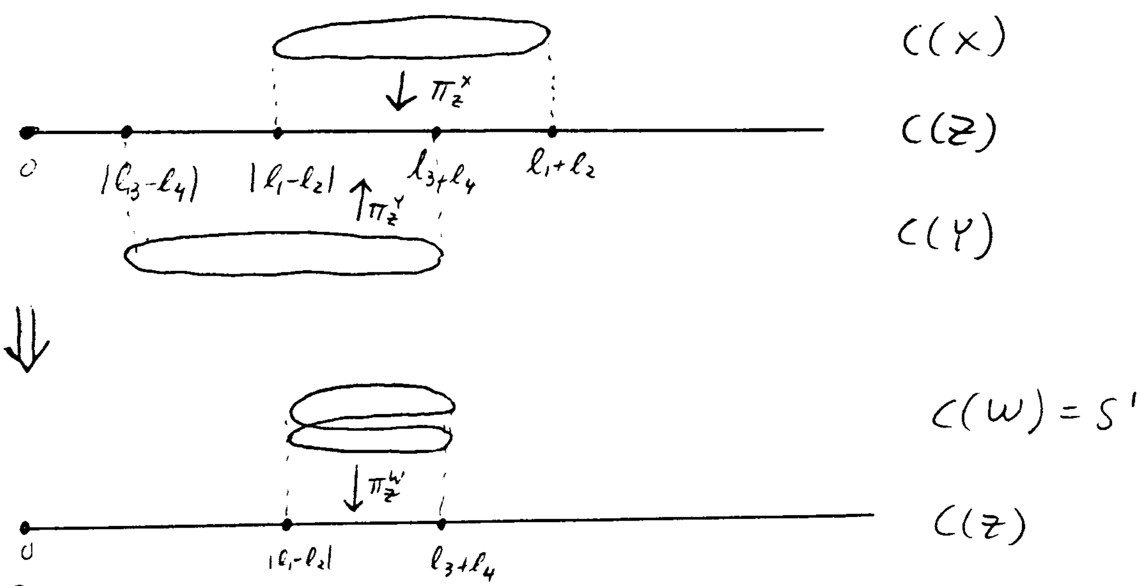
So $C(L') = C(L \bigcup E) = C(L) \times S^1$. ⊠

Examples. (a)

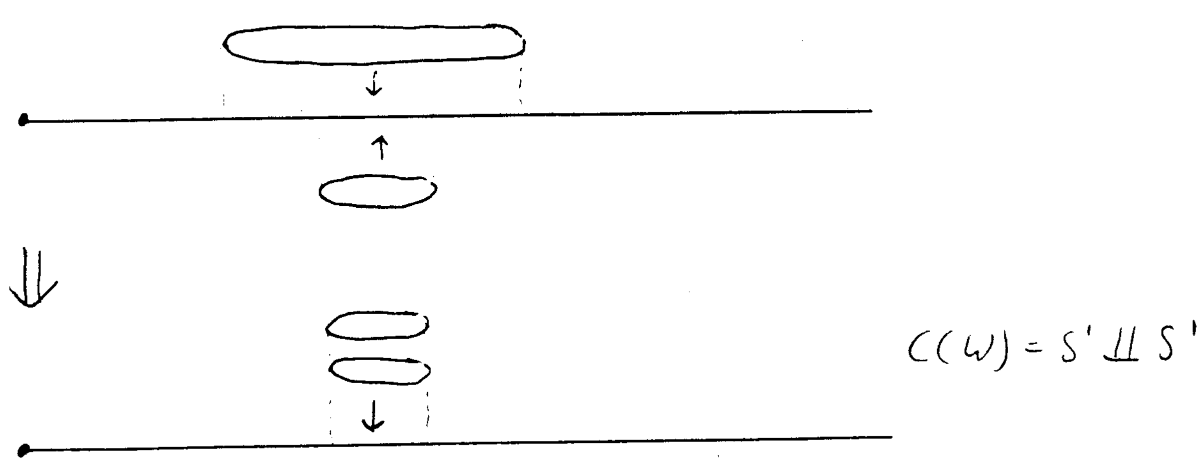


$C(X) = C(Y) = S^1$, $C(Z) = \mathbf{R}_+$. If $l_1 \neq l_2$ or $l_3 \neq l_4$, then then points of Z never coincide $\Rightarrow$ stab(z) = 1 for all $e \in F(Z) \Rightarrow C^*(Z) = C(Z)$.

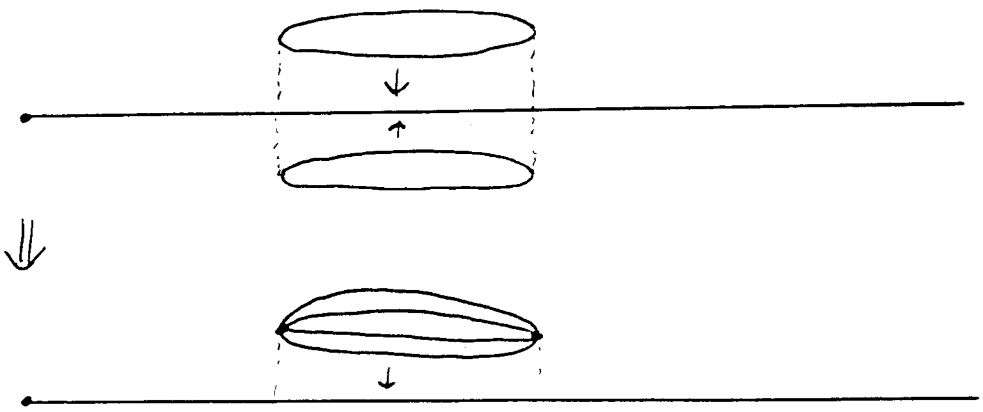
If $l_1+l_2 > l_3+l_4 > |l_1-l_2| > |l_3-l_4|$,



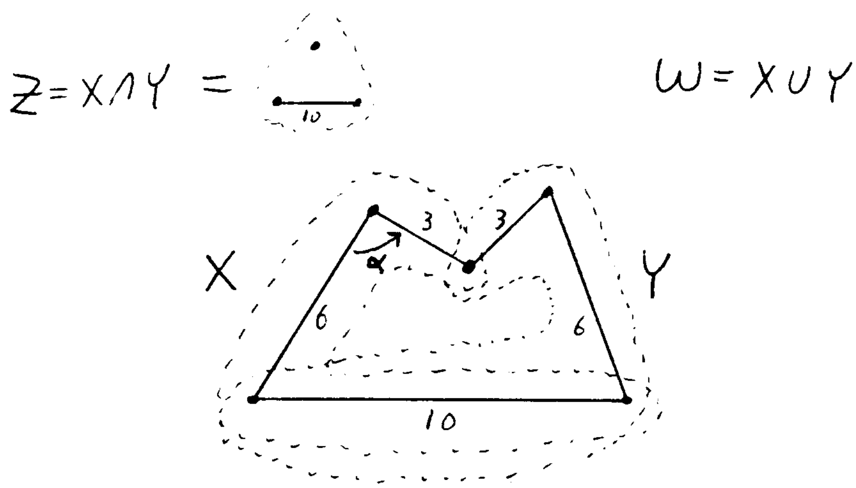
Other possibilities are



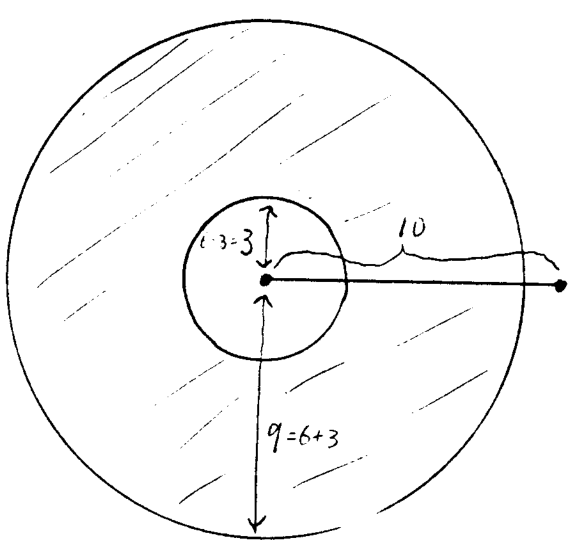
and



(b)



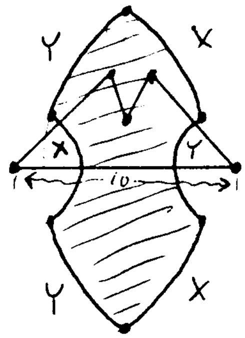
$C(X) = C(Y) = S^1 \times S^1$, $C(Z) = \mathbf{R}^2$. Identify $C(Z)$ with the plane of the diagram. Holding the bar of length 10 fixed, we can parameterize $C(X)$ by the position of the vertex v and $\mathbf{Z}_2$ = (whether α is $< \pi$ or $> \pi$). $C(X)$ projects to $C(Z) = \mathbf{R}^2$ as follows



$(\pi^X_Z)^{-1}$(interior pt) $= 2$ points

$(\pi^X_Z)^{-1}$(boundary pt.) $= 1$ point

The same goes for $C(Y)$. $\pi_Z^X(C(X)) \bigcap \pi_Z^Y(C(Y))$ is

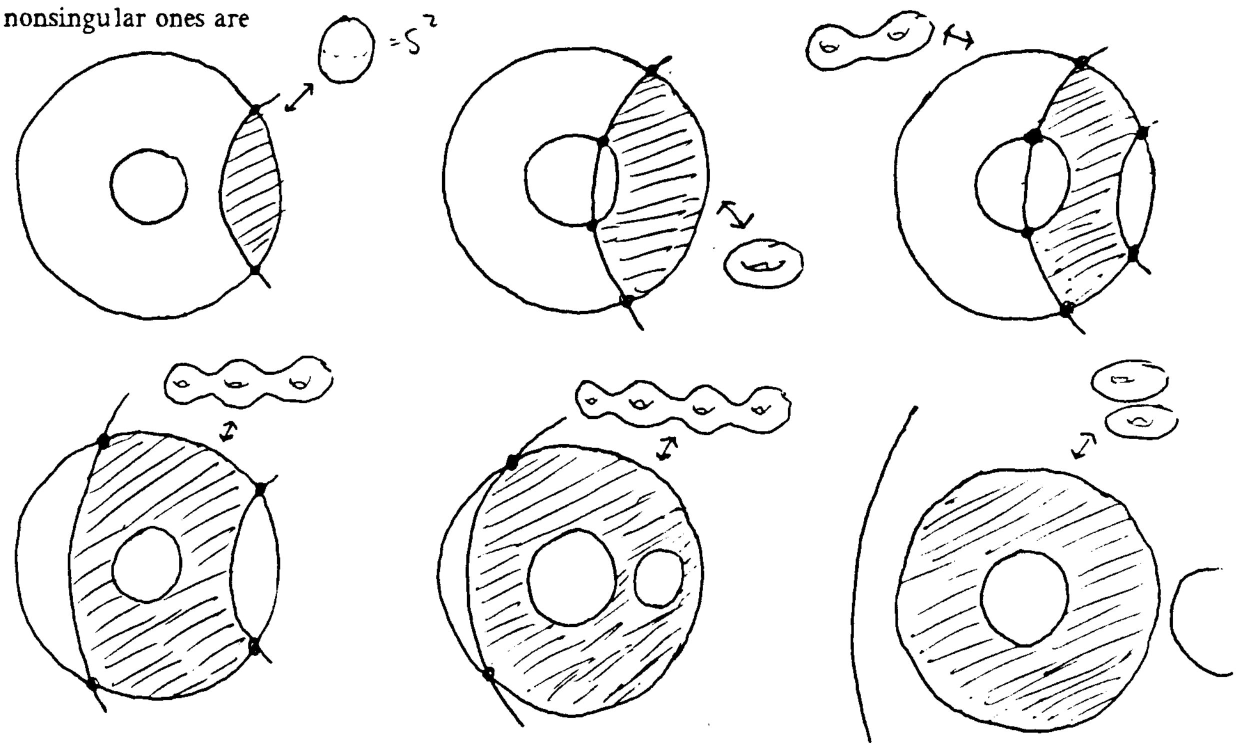


Above each each interior point of the hexagon, there are $\mathbf{Z}_2 \times \mathbf{Z}_2 = 4$ points. The first $\mathbf{Z}_2$ factor is associated with X and the second with Y. On the sides labeled X the first $\mathbf{Z}_2$ collapses to a point ($\alpha = 0$ or π). On the sides labeled Y, the second $\mathbf{Z}_2$ collapses. In effect we have four copies of the hexagon, labeled 00, 01, 10, 11. On the X-sides [Y-sides], hexagons with the same second [first] digit are identified. It is easy to see that this gives an orientable 2-manifold. The Euler characteristic is

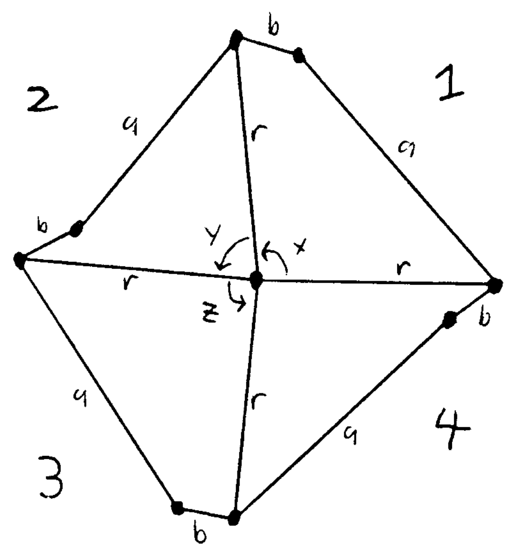
$$\text{faces - edges + vertices} = 4 - (6\ 4)/2 + (6\ 4)/4 = -2,$$

so the genus is $1 - (-2)/2 = 2$. (See [Thurston-Weeks].)

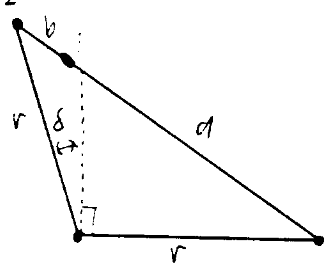
This method can be used to enumerate all configuration spaces of the pentagon $C(\Sigma_5, l)$. The nonsingular ones are

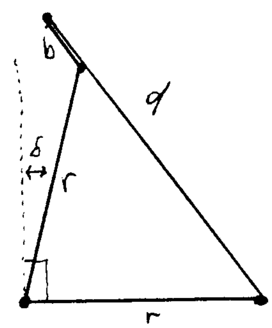


(c)

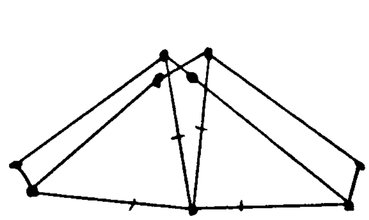


Let a and b be such that the maximum of x is $\frac{\pi}{2}+\delta$ and the minimum of x is $\frac{\pi}{2}-\delta$, where δ is, say, $\frac{\pi}{12}$.

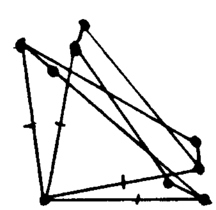




Parameterize $C(L)$ by x, y and z. We consider only the component of $C(L)$ containing the position shown above. (Not, for example,



or



Each elbow on the perimeter gives rise to an inequality on x, y, z:

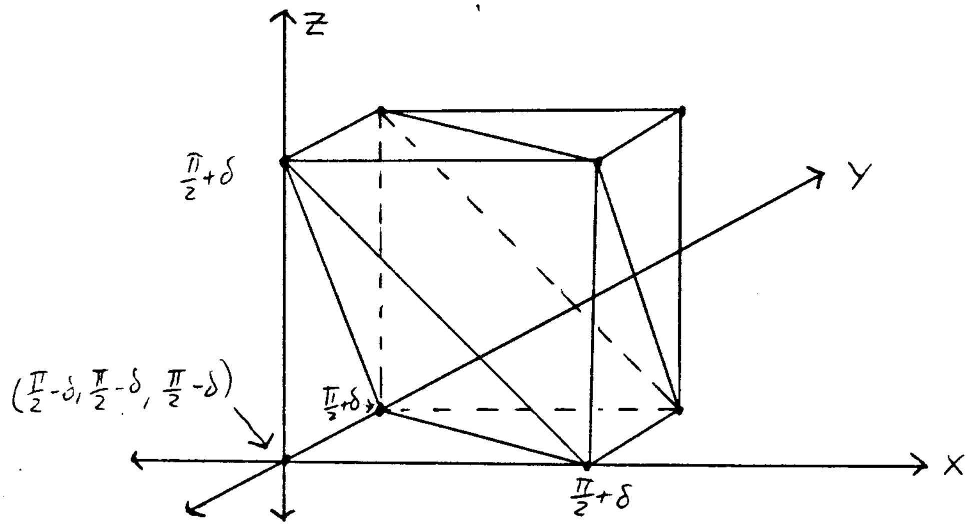
(1) $\frac{\pi}{2}-\delta \leqslant x \leqslant \frac{\pi}{2}+\delta$

(2) $\frac{\pi}{2}-\delta \leqslant y \leqslant \frac{\pi}{2}+\delta$

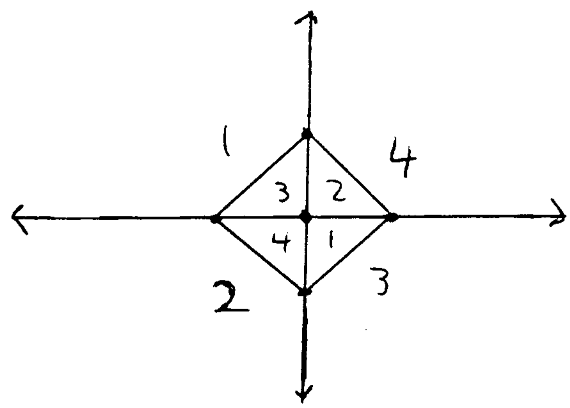
(3) $\frac{\pi}{2}-\delta \leqslant z \leqslant \frac{\pi}{2}+\delta$

(4) $\frac{\pi}{2}-\delta \leqslant 2\pi-(x+y+z) \leqslant \frac{\pi}{2}+\delta$

The set of $(x,y,z) \in \mathbf{R}^3$ satisfying these inequalities is an octahedron



Each (x,y,z) in the interior corresponds to 2^4 points in $C(L)$; each elbow can be in two positions. On the faces corresponding to elbow i, these two positions degenerate into one. So, as in the previous example, $C(L)$ consists of 16 octahedra labeled like this

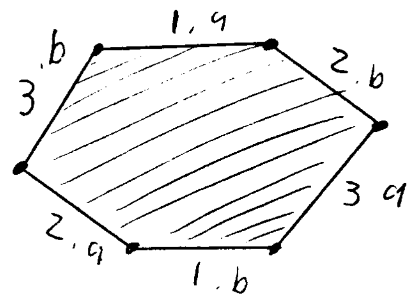


and indexed by a 4 digit binary number. On faces labeled i, octahedra whose indices differ only in the i^{th} digit are glued together via the identity map.

All 16 octahedra meet at each vertex, and the cone over a vertex is $S^1 \times S^1$. So $C(L)$ is a cone manifold with 6 singular points.

We can use a standard trick to give $C(L)$ a hyperbolic structure. Put the octahedron in H^3 (hyperbolic 3-space) and expand it until the dihedral angles are $\frac{\pi}{2}$ (4 octahedra meet at each edge).

As in (b),



gives a surface G_3 of genus 3 (See [Thurston-Weeks]). The sides marked "a" correspond to three disjoint closed curves in G_3. The same goes for the b-sides. The nonsingular part of $\mathcal{C}(L)$ is homeomorphic to $G_3 \times S^1$ - (tubular nbd of (a-curves, 0) $\bigcup$ tubular nbd of (b-curves, π)).

§ 3 Cyclic Linkages

In this section we treat cyclic linkages in detail. We begin by finding the critical points of $p_n := p_{\Sigma_n} : \mathbf{CP}^{n-2} \to \Delta^{n-1}$. Next we relate the configuration spaces corresponding to regular values which are near each other. We use this relation to calculate the homology of $C(\Sigma_n, l)$. We also calculate the ring structure of $H^*(C(\Sigma_n, l))$, modulo a technical lemma.

All coefficients for (co)homology are in $\mathbf{Z}$.

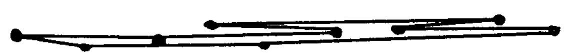
We sometimes write L for $C(\Sigma_n, l)$.

For each $A \subset \{1, 2, \ldots, n\}$ let P_A be the n-2 plane

$$\sum_{i \in A} t_i = \sum_{i \notin A} t_i = 1/2$$

in Δ^{n-1}.

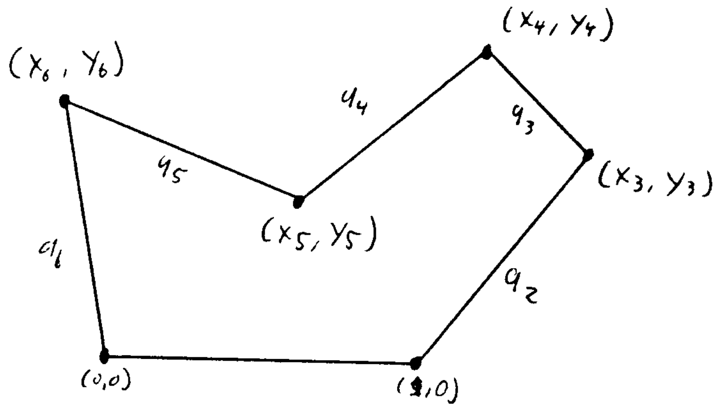
Proposition 3.1. The critical values of $p_n : \mathbf{CP}^{n-2} \to \Delta^{n-1}$ in the interior of Δ^{n-1} are the planes P_A, for all $A \neq \emptyset, \{1, 2, \ldots, n\}$. $c \in C(\Sigma_n, l)$ is a critical point if and only if $\alpha(e_i, e_j)(c) = 0$ or π for all i, j; that is, c is "flat."



Proof. (Due to Thurston.) Let $V = \{(t_i) \in \Delta^{n-1} : t_1 \neq 0\}$, $W = \{(v_i) \in (\mathbf{R}^2)^n : v_1 = (0,0), v_2 = (1,0)\}$, $U = \{(a_i) \in \mathbf{R}^n_+ : a_1 = 1\}$. Then

$$\begin{tikzcd}
(\mathbb{R}^2)^n \supset W \arrow[r, "\sim"', "/\Gamma \times \mathbb{R}_+"] \arrow[d, "\tilde{p}_n|_W"] & p_n^{-1}(V) \arrow[d, "p_n"] \\
\mathbb{R}^n_+ \supset U \arrow[r, "\sim", "/\mathbb{R}_+"'] & V \subset \Delta^{n-1}
\end{tikzcd}$$

commutes, and the horizontal maps are homeomorphisms. Also, interior$(\Delta^{n-1}) \subset V$. So we prove the result for $\tilde{p} := \tilde{p}_{\Sigma_n}|_W : W \to U$.

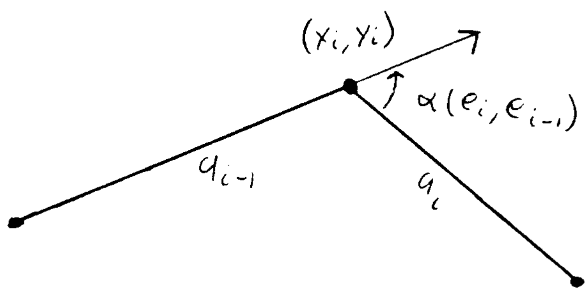


$(x_3,y_3), \ldots, (x_n,y_n)$ are coordinates on W, and $a_2, \ldots, a_n$ are coordinates on U. Let $v = ((x_3,y_3), \ldots, (x_n,y_n)) \in W$. Let d_i, $3 \leqslant i \leqslant n$, be the unit vector in the direction

$$(x_i - x_{i-1})\frac{\partial}{\partial x_i} + (y_i - y_{i-1})\frac{\partial}{\partial y_i}$$

Then

$$D\tilde{p}(v)(d_i) = \frac{\partial}{\partial a_{i-1}} - \cos(\alpha(e_i,e_{i-1})(v))\frac{\partial}{\partial a_i}$$

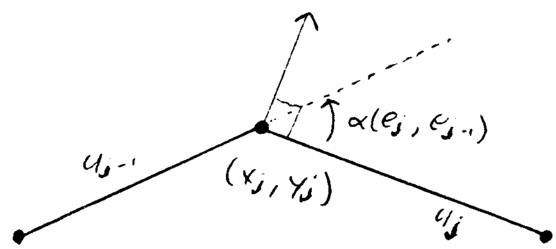


This gives n-2 linearly independent vectors in $D\tilde{p}(v)(v)(TW_v) \subset TU_{\tilde{p}(v)}$. If $\alpha(e_j,e_{j-1})(v) \neq 0, \pi$, then

$$D\tilde{p}(v)(c_j) = \sin(\alpha(e_j,e_{j-1})(v))\frac{\partial}{\partial a_{j-1}}$$

where c_j is the unit vector in the direction

$$(y_{j+1}-y_j)\frac{\partial}{\partial x_j} - (x_{j+1}-x_j)\frac{\partial}{\partial y_j}$$



This gives an $(n-1)^{th}$ linearly independent vector in $D\tilde{p}(v)(TW_v)$, so v is a regular point in this case.

If $\alpha(e_j, e_{j-1})(v) = 0$ or π for all j, then the d_i's and the c_i's span TW_v, and we see that $\dim(D\tilde{p}(v)(TW_v)) = n\text{-}2 < n\text{-}1 = \dim TU_{\tilde{p}(v)}$.

So Δ^{n-1} is divided into cells by the planes P_A, $A \subset \{1, 2, \ldots, n\}$. The interiors of the cells are regular values, and hence the corresponding configuration spaces are manifolds.

Proposition 3.2. Suppose $l = (t_i)$ and $l' = (t'_i)$ lie in the same cell. That is, for all $A \subset \{1, 2, \ldots, n\}$,

$$\sum_{i \in A} t_i \neq {}^1\!/_2 \neq \sum_{i \in A} t'_i$$

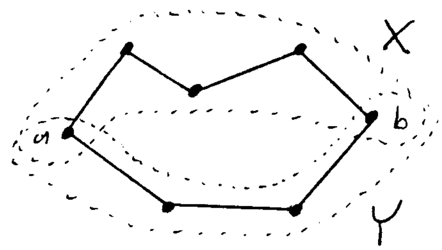
and

$$\sum_{i \in A} t_i > {}^1\!/_2 \Leftrightarrow \sum_{i \in A} t'_i > {}^1\!/_2$$

Then $C(\Sigma_n, l) \cong C(\Sigma_n, l')$.

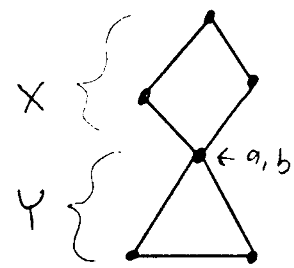
Proof. Let S be the line segment in Δ^{n-1} between l and l'. Since the cells are convex, all points in S are regular values. Let $\lambda : S \to [0,1]$, $\lambda(l) = 0$, $\lambda(1) = 1$ be linear. Consider $W := p_n^{-1}(S)$ and $\lambda' := \lambda p_n : W \to [0,1]$. W is a manifold with boundary components $\lambda'^{-1}(0) = C(\Sigma_n, l)$ and $\lambda'^{-1}(1) = C(\Sigma_n, l')$, and λ' is a Morse function without critical points. Therefore $W \cong [0,1] \times C(\Sigma_n, l) \cong [0,1] \times C(\Sigma_n, l')$ and $C(\Sigma_n, l) \cong C(\Sigma_n, l')$ (see [Milnor, p. 12]).

Remark. If we divide Σ_n into two chains



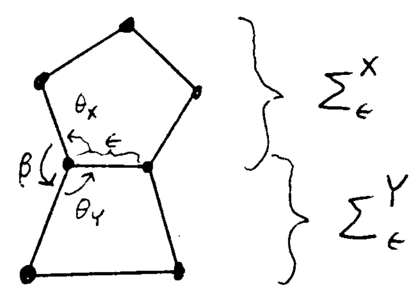
$X \cap Y = \{a, b\}$

and apply 2.5, we must ignore positions where points a and b coincide.



For then stab({a,b}) = $S^1 \neq$ id. We can now patch this hole up.

Let j be the number of bars in X and k be the number of bars in Y. Let Σ^X and Σ^Y be the linkages obtained by joining the ends of X and Y, respectively. Let Σ^X_ϵ be the j+1 bar cyclic linkage obtained by inserting a bar of length ϵ between the endpoints of X, and define Σ^Y_ϵ similarly.



Note that $(\pi^{\Sigma_n}_{X\cap Y})^{-1} = C(\Sigma^X_\epsilon) \times C(\Sigma^Y_\epsilon)$.

Assume that Σ^X and Σ^Y correspond to points on the interiors of cells in Δ^{j-1} and Δ^{k-1}. Then, by corollary 2.5 and proposition 3.2, there is a δ such that for $0 < \epsilon < \delta$, $C(\Sigma^X_\epsilon) \cong C(\Sigma^X) \times S^1$ and $C(\Sigma^Y_\epsilon) \cong C(\Sigma^Y) \times S^1$, so $(\pi^{\Sigma_n}_{X\cap Y})^{-1}(\epsilon) = C(\Sigma^X) \times C(\Sigma^Y) \times S^1 \times S^1$ and $(\pi^{\Sigma_n}_{X\cap Y})^{-1}((0,\delta)) = C(\Sigma^X) \times C(\Sigma^Y) \times S^1 \times S^1 \times (0,\delta)$.

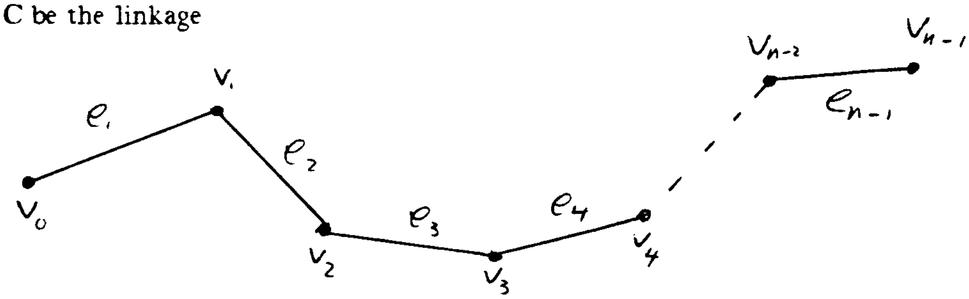
On the other hand $(\pi^{\Sigma_n}_{X\cap Y})^{-1}(0) \cong C(\Sigma^X) \times C(\Sigma^Y) \times S^1$. How are these two pieces of $C(L)$ connected?

The projection $(\pi^{\Sigma_n}_{X\cap Y})^{-1}((0,\delta)) = C(\Sigma^X) \times C(\Sigma^Y) \times S^1 \times S^1 \to S^1 \times S^1$ is given by $x \to (\theta_X(x), \theta_Y(x))$, and the projection $(\pi^{\Sigma_n}_{X\cap Y})^{-1}(0) = C(\Sigma^X) \times C(\Sigma^Y) \times S^1 \to S^1$ is $x \to \beta(x)$. It is clear from the diagram that $\beta+(\theta_X+\theta_Y) = 2\pi$. So $((a, b, \theta_X, \theta_Y, \epsilon) \in (\pi^{\Sigma_n}_{X\cap Y})^{-1}((0,\delta))) \to ((a, b, -(\theta_X+\theta_Y)) \in (\pi^{\Sigma_n}_{X\cap Y})^{-1}(0))$ as $\epsilon \to 0$. The cycles $\theta_X+\theta_Y$ = const collapse to points.

Now we relate the topology of configuration spaces of adjacent cells.

Let $\Phi \subset P(\{1, 2, \ldots, n\})$ be such that $A \in \Phi \Leftrightarrow \bar{A} \notin \Phi$. That is, Φ contains one representative of each pair of complementary subsets of $\{1,2, \ldots, n\}$. Then each cell E corresponds to a unique subset I of Φ, where $A \in I$ iff $\sum_{i \in A} t_i < 1/2$ for $(t_i) \in E$. Call I the index of E. Adjacent cells have indices which differ by only one element.

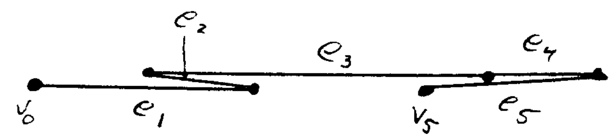
Let C be the linkage



where $l(e_i) = l_i$, and let S be the sublinkage consisting of the endpoints. $C(C) = T^{n-2}$ and $C(S) = \mathbf{R}_+$. Parameterize $C(S)$ by $|v_{n-1} - v_0|$. Consider the function $\pi^C_S : C(C) \to C(S) \subset \mathbf{R}$. $(\pi^C_S)^{-1}(d) \cong C(\Sigma_n, \{l_1, \ldots, l_n, d\})$.

Assume that $l_i > 0$, $1 \leq i \leq n-1$, and that $\pi^C_S(x) > 0$. If one of the points $v_1, \ldots, v_{n-2}$ is not colinear with v_0 and v_{n-1}, then varying the angle at that point shows that x is not a critical point of π^C_S. Thus the only possible critical points are those where all the vertices are colinear = those where $\alpha(e_i,e_{i+1}) = 0$ or π, $1 \leq i \leq n-2$.

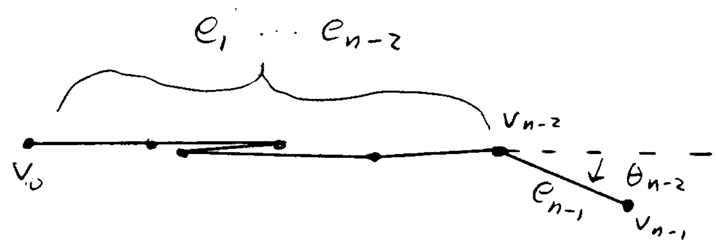
Let β_i denote the angle from $e_i = (v_{i-1},v_i)$ to the directed line (v_0,v_{n-1}). For $A \subset \{1, 2, \ldots, n-1\}$, let x_A be the unique (if it exists) element of $C(C)$ such that $\beta_i = \pi$ if $i \in A$ and $\beta_i = 0$ otherwise.



$X_{\{2,5\}}$

Proposition 3.3. For each $A \subset \{1, 2, \ldots, n\}$ such that x_A exists, x_A is a nondegenerate critical point of π_S^C of index $|A| - 1$.

Proof. This is clear for $n = 3$. Assume it holds for $n = m-1$ and consider the case $n = m$. I'll prove the case where $n-1 \in A$ and the angle between (v_0, v_{n-2}) and (v_{n-2}, v_{n-1}) is 0. The other cases are similar.



By the law of cosines,

$$|v_{n-1}-v_0|^2 = |v_{n-2}-v_0|^2 + l_{n-1}^2 + 2|v_{n-2}-v_0| l_{n-1} \cos(\theta_{n-2})$$

Let C' be the sublinkage consisting of $e_1, \ldots, e_{n-2}$. By the inductive hypothesis (and the Morse lemma) there are coordinates $\bar{\theta} = (\theta_1, \ldots, \theta_{n-3})$ on a neighborhood of x_A in $C(C')$ such that

$$\begin{aligned} |v_{n-2}-v_0| &= d - \sum_{i=1}^{|A'|} \theta_i^2 + \sum_{i=|A'|+1}^{n-3} \theta_i^2 \\ &=: d + Q(\bar{\theta}). \end{aligned}$$

$\theta_1, \ldots, \theta_{n-2}$ are coordinates in a neighborhood of x_A in $C(C)$. To second order,

$$\begin{aligned} |v_{n-1}-v_0|^2 &= (d+Q(\bar{\theta}))^2 + l_{n-1}^2 + 2(d+Q(\bar{\theta}))l_{n-1}\left(1 - \frac{\theta_{n-2}^2}{2}\right) \\ &= (d+l_{n-1})^2 + 2(d+l_{n-1})Q(\bar{\theta}) - d\theta_{n-2}^2 \end{aligned}$$

$$|v_{n-1}-v_0| \;=\; (d+l_{n-1}) + \frac{1}{d+l_{n-1}}Q(\bar{\theta}) - \frac{d}{(d+l_{n-1})^2}\theta_{n-2}^{2}. \qquad \boxtimes$$

Thus the effect of adding $A \subset \{1, 2, \ldots, n\}$ to the index of a configuration space is the same as the effect of passing through a critical point of index $|A| - 1$ (in the negative direction) on the level surfaces of a Morse function. That is, the configuration spaces corresponding to the indices $\{A_1, \ldots, A_p\}$ and $\{A_1, \ldots, A_p, A\}$ differ by a surgery of the form $S^j \times D^{k+1} \to D^{j+1} \times S^k$ where $j = |A| - 2$ and $j + k + 1 = n - 3 = \dim C(\Sigma_n, l)$.

We will now compute the homology of $C(\Sigma_n, l)$.

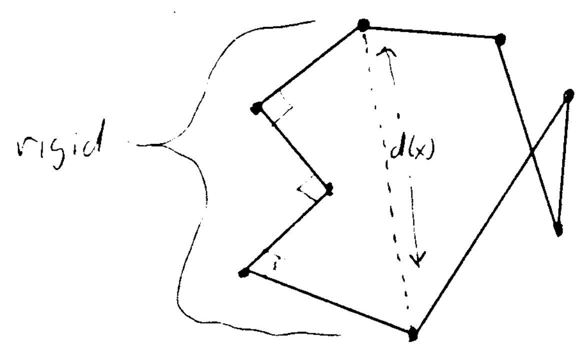
Consider in general a surgery of this form on a manifold M yielding a manifold M'. Let $\overline{M} = M - \mathrm{int}(S^j \times D^{k+1}) = M' - \mathrm{int}(D^{j+1} \times S^k)$. Let $S = S^j \times S^k = \partial\overline{M} = \partial(S^j \times D^{k+1}) = \partial(D^{j+1} \times S^k)$. Suppose $S^j \times pt \subset S \subset \overline{M}$ bounds in $\overline{M}$. Then, by standard results in Morse theory,

$$H_i(M') \;=\; \begin{cases} H_i(M) \;\; \mathbf{Z}, & i = m, n-1 \\ \\ H_i(M), & \text{otherwise} \end{cases}$$

Cycles which represent generators of the new stuff in $H_k(M')$ and $H_{j+1}(M')$ are $pt \times S^k \subset S \subset M'$ and $B \bigcup (D^{j+1} \times pt)$, where B is a chain in $\overline{M}$ such that $\partial B = S^j \times pt \subset S \subset \overline{M}$ and $D^{j+1} \times pt \subset D^{j+1} \times S^k \subset M'$. If there are cycles in $\overline{M}$ which represent generators of $H_*(M)$, these same cycles (in M') represent generators of the corresponding parts of $H_*(M')$.

Before applying the above to the case at hand, we will define some cycles (actually submanifolds) of $C(\Sigma_n, l)$.

Let $A \subset \{1, 2, \ldots, n\}$. Fixing the angles between the edges of A defines a submanifold (usually). Without loss of generality, the edges of A are adjacent (corollary 2.2). Let X be the sublinkage of $C(\Sigma_n, l)$ consisting of the edges of A.



Let $x \in X$. $(\pi_X^1)^{-1}(x) \subset C(\Sigma_n, l)$ is homeomorphic to $C(\Sigma_{n-k}, l')$, where l' has the lengths of l not in A plus $d(x) :=$ distance between the end points of X. $(\pi_X^1)^{-1}(x)$ is generically a submanifold (proposition 3.1).

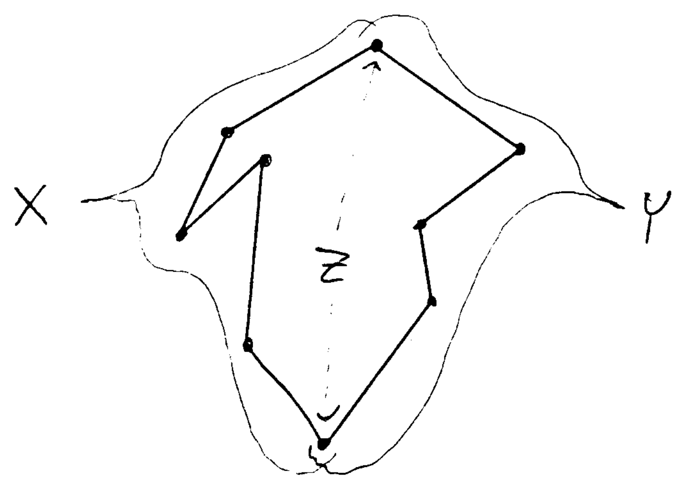
The homology class $r(A) := [(\pi_X^1)^{-1}(x)]$ is independent of x. (We will ignore signs.) For, given $x, y \in C(X) = T^{|A|-1}$, we can find a path $\gamma : [0,1] \to C(X)$, $\gamma(0) = x$, $\gamma(1) = y$, such that $d\,\gamma : [0,1] \to \mathbf{R}_+$ is monotone. Then $(\pi_X^1)^{-1}(\gamma([0,1]))$ gives a cobordism of $(\pi_X^1)^{-1}(x)$ and $(\pi_X^1)^{-1}(y)$ (see 3.3).

If there is a section

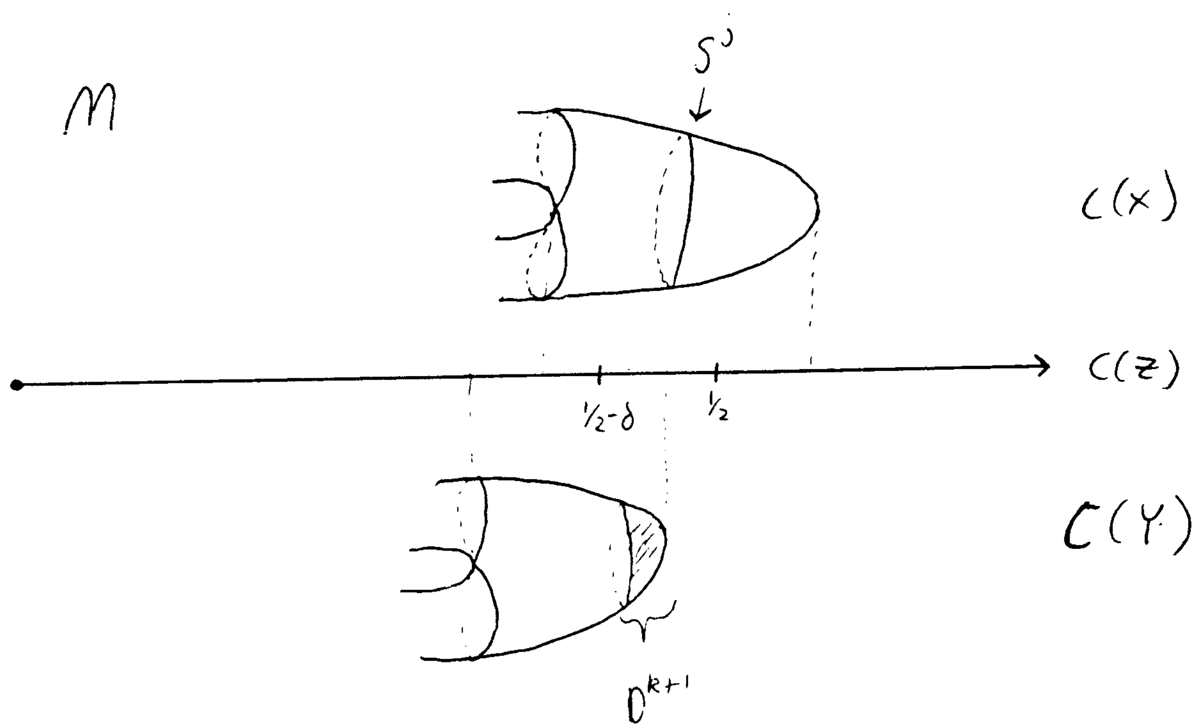
$$s_A : C(X) \to C(L)\ , \qquad \pi_X^1 s_A = \mathrm{id}$$

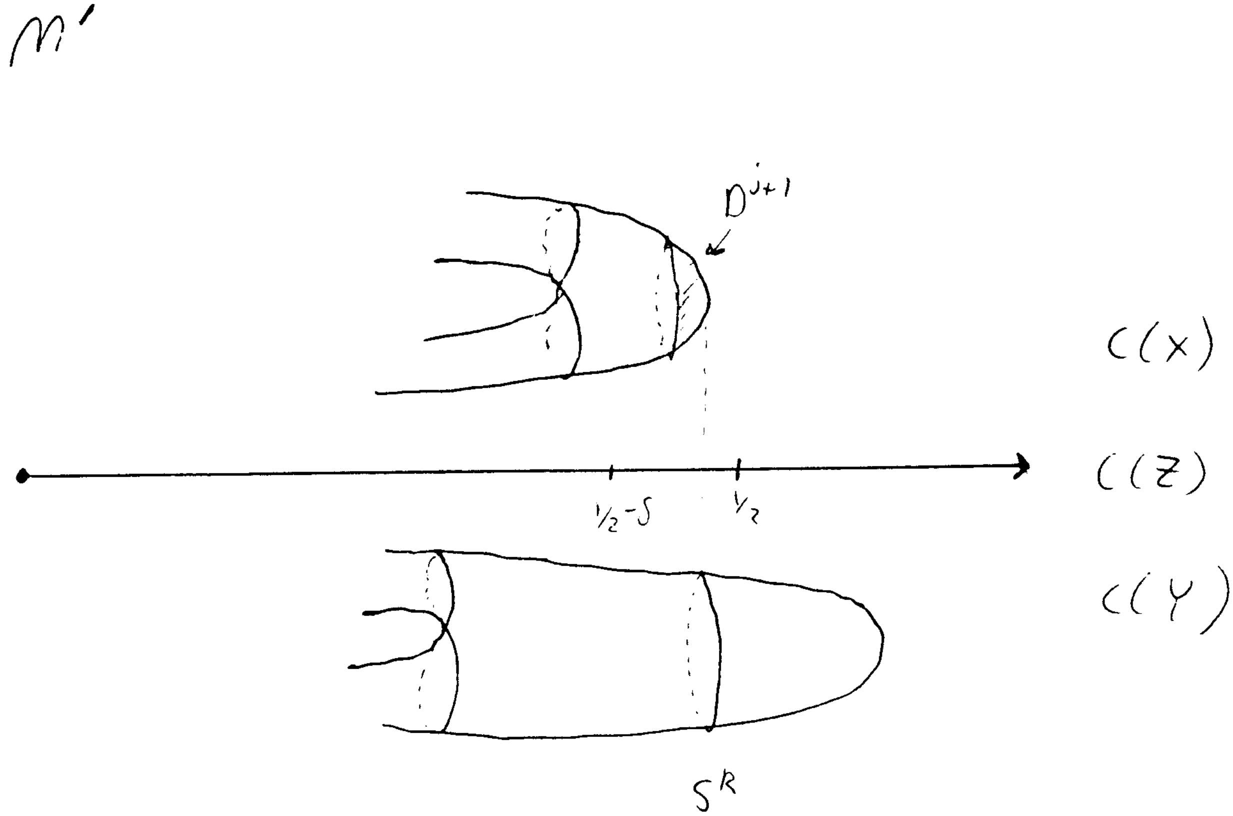
define $f(A, s_A) = [s_A(C(X))] \in H_{|A|-1}(C(L))$.

Now consider adjacent cells with indices $\{B_1, \ldots, B_j\}$ and $\{B_1, \ldots, B_j, A\}$. Let M and M' be the corresponding configuration spaces. By 3.3, M' is obtained from M by a surgery of the form $S^j \times D^{k+1} \to D^{j+1} \times S^k$, where $j = |A|-2$, $k+j+1 = n-3 = \dim M$. Let $A = \{i_0, \ldots, i_j\}$. Assume (W.L.O.G.) that these edges are adjacent on Σ_n.



Let X be the sublinkage of Σ_n consisting of the edges in A, and let Y be the sublinkage consisting of the remaining edges. $X \cap Y = 2$ points $=: Z$. $C(Z) = \mathbf{R}_+$. By 2.4, M (or M') is the fibered product of $\pi_Z^X : C(X) \to \mathbf{R}_+$ and $\pi_Z^Y : C(Y) \to \mathbf{R}_+$. (This isn't quite true, but see the remark following 3.2.) How does this product differ for indices $\{B_i\}$ and $\{B_i, A\}$? By 3.3 (and the preceeding discussion), we see that the only difference is above a neighborhood of $\frac{1}{2} \in \mathbf{R}_+$.


M
S^j
$c(X)$
$c(Z)$
$1/2-\delta$
$1/2$
$C(Y)$
D^{k+1}


M'
D^{j+1}
$c(X)$
$c(Z)$
$1/2-\delta$
$1/2$
$c(Y)$
S^k

For some $\delta > 0$, the part of M lying above $[0,\frac{1}{2}-\delta]$ is homeomorphic to the corresponding part of M'. However, $(\pi_Z^M)^{-1}([\frac{1}{2}-\delta,\infty))$ is homeomorphic to $S^j \times D^{k+1}$, while $(\pi_Z^{M'})^{-1}([\frac{1}{2}-\delta,\infty))$ is homeomorphic to $D^{j+1} \times S^k$. $(\pi_Z^M)^{-1}(\frac{1}{2}) = (\pi_Z^{M'})^{-1}(\frac{1}{2}) = S^j \times S^k$. **We have found the surgery!** Reviving the notation of above, $S = (\pi_Z^M)^{-1}(\frac{1}{2})$, $\overline{M} = (\pi_Z^M)^{-1}([0,\frac{1}{2}-\delta])$. $pt \times S^k \subset S$ corresponds to holding X rigid and letting Y wiggle; it is homologous to $r(A)$. Similarly, $S^j \times pt$ is homologous to $r(\overline{A})$.

Note that in M, $pt \times S^k$ can be shrunk to a point

and in M', $S^j \times pt$ can be shrunk to a point

as should be the case.

By choosing different points in the same cells (and relabeling if necessary), we may assume that the linkages have the same longest edge. Assume this edge is in A. It will be shown below that under these conditions there is a section $s_A : C(X) \to C(\Sigma_n,\{B_i, A\})$ of π_X^1. It is easily seen that $s_A(C(X)) \bigcap \overline{M}$ is a cycle whose boundary is $S^j \times pt$. Thus the remarks above apply to this case and we get

Proposition 3.4. If A contains the longest edge (which may be assumed to exist), then

$$H_i(C(\Sigma_n,\{B_i, A\})) = \begin{cases} H_i(C(\Sigma_n,\{B_i\}))\oplus \mathbf{Z}, & i = (n-3)-k,\ k \\ H_i(C(\Sigma_n,\{B_i\})), & \text{otherwise} \end{cases}$$

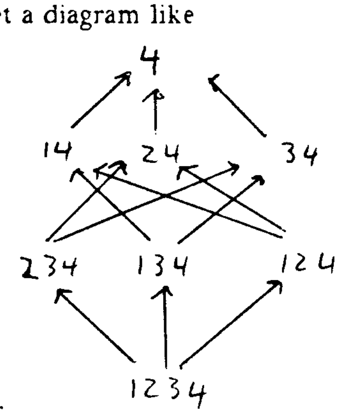
and $r(A)$ and $f(A,s_A)$ are generators of the new stuff of $H_*(C(\Sigma_n, \{B_i, A\}))$.

So to compute the homology of any cell, we can start with a cell of known homology and move from cell to cell, making sure the transitions are of the above type. The easiest way to do this is to keep the lengths of the smaller sides fixed (actually their ratios fixed; $\sum$ sides = 1) while shrinking the longest side from a very long length to it's final length.

Let m denote the longest edge. Let $\Phi_n = \{A \subset P(\{1, 2, \ldots, n\}) : m \in A\}$. If m is very long (i.e. $> 1/2$), then Σ_n cannot close, so $C(L) = \emptyset$ and $H_*(C(L)) = 0$. The index of this cell is $\emptyset \subset \Phi_n$. As m shrinks, the next cell we come to has index $\{\{m\}\}$. The configuration space of this cell is obtained from $\emptyset$ by $S^{-1} \times D^{(n-3)+1} \to D^0 \times S^{n-3}$, so the configuration space of this cell is S^{n-3}. A generator of $H_{n-3}(C(L))$ is $r(\{m\}) = [C(L)]$ and a generator of $H_0(C(L))$ is $f(\{m\},s_m) = [pt]$. We proceed in this manner, adding homology generated by $r(A)$ and $f(A,s_A)$ (for some section s_A) each time we add a subset $A \subset \{1, 2, \ldots, n\}$ to the index. At each transition $\{B_i\} \to \{B_i, A\}$, it is possible to choose representatives of $r(B_i)$ and $f(B_i, S_{B_i})$ disjoint from the surgery region. (This is easy for the r's; for the f's it follows from the construction of the sections given below.) So the cycles continue to represent the appropriate homology classes. In summary,

Proposition 3.5. The homology of $C(\Sigma_n,l)$ is freely generated by $\{r(A)\}$ and $\{f(A,s_A)\}$, where A ranges over all subsets of $\{1, 2, \ldots, n\}$ such that the longest edge is in A and $\Sigma A < 1/2$.

Examples. (a) Σ_4. Label the edges by 1, 2, 3, 4. Write abc for $\{a, b, c\}$. $\Phi_4 = \{4, 14, 24, 34, 124, 134, 234, 1234\}$. Not all subsets of Φ_4 are possible indices. If $A \subset B$, $\Sigma B < 1/2 \Rightarrow \Sigma A < 1/2$. Taking this into account, we get a diagram like



If $I \subset \Phi_4$ is an index and $A \in I$, then all subsets "above" A in the diagram must also be in I.

W.L.O.G., $l(1) \leqslant l(2) \leqslant l(3) < l(4)$. This gives some more constraints on indices. For example, $\Sigma 24 < \frac{1}{2} \Rightarrow \Sigma 14 < \frac{1}{2}$. We get a new diagram:

$$\begin{tikzcd}
 & & & 4 \\
 & 34 \arrow[r] & 24 \arrow[r] & 14 \arrow[u] \\
234 \arrow[r] & 134 \arrow[r] \arrow[u] & 124 \arrow[u] & \\
1234 \arrow[u] & & &
\end{tikzcd}$$

Finally, we note that $\Sigma A + \Sigma \bar{A} = 1$ for all $A \subset \{1, 2, \ldots, n\}$. This implies that 24, and hence all the subsets below it, can never be in the index of a cell (assuming $l(1) \leqslant l(2) \leqslant l(3) < l(4)$). So the only significant part of the diagram is

$$\begin{tikzcd}
4 \\
14 \arrow[u]
\end{tikzcd}$$

and the possible indices are $\varnothing$, $\{4\}$ and $\{4, 14\}$. By propsition 3.5., $C(\varnothing) = \varnothing$, $C(\{4\}) = S^1$, and $C(\{4, 14\}) = S^1 \bigcup S^1$.

(b) Σ_5. Label the edges by 1, 2, 3, 4, 5, and assume $l(1) \leqslant l(2) \leqslant l(3) \leqslant l(4) < l(5)$. Proceeding as before, we end up with this this diagram

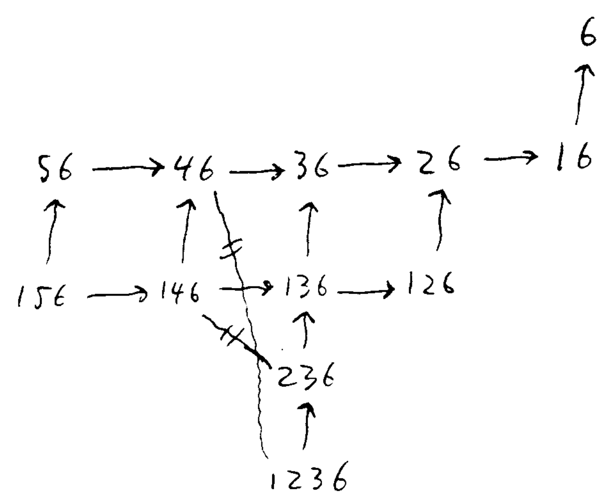
$$\begin{tikzcd}
 & & & 5 \\
45 \arrow[r] & 35 \arrow[r] \arrow[dr, dash, "/\!/" description] & 25 \arrow[r] & 15 \arrow[u] \\
 & & 125 \arrow[u] &
\end{tikzcd}$$

"$\not\!\!\!\longrightarrow$" means that ~~45~~ 35 and 125 cannot both be in an index. For $\Sigma \cancel{45}\,35 < \frac{1}{2} \Rightarrow \Sigma 12\cancel{3}\,4 > \frac{1}{2} \Rightarrow \Sigma 125 > 1/2$, and $\Sigma 125 < \frac{1}{2} \Rightarrow \Sigma 34 > \frac{1}{2} \Rightarrow \Sigma \cancel{45}\,35 > 1/2$. The possible indices are $\varnothing$, $\{5\}$, $\{5, 15\}$, $\{5, 15, 25\}$, $\{5, 15, 25, 35\}$, $\{5, 15, 25, 35, 45\}$ and $\{5, 15, 25, 125\}$. By 3.5.,

$C(\varnothing) = \varnothing$
$C(\{5\}) = S^2$
$C(\{5,15\}) = G_1$
$C(\{5,15,25\}) = G_2$
$C(\{5,15,25,35\}) = G_3$
$C(\{5,15,25,35,45\}) = G_4$
$C(\{5,15,\underset{25}{\wedge}125\}) = G_1 \cup G_1$

where G_i is the orientable surface of genus i. This agrees with computation of the configuration spaces of Σ_5 given in §2.

(c) Σ_6. In this case, the diagram is



Things are more difficult this time, since a 3-manifold is not determined by it's homology. If one of the bars is very small, we know the configuration space by 2.5 and the previous example. It turns out that most of the indices differ from one of these "known" indices by subsets of size 2, so most of the configuration spaces can be obtained from known ones by $S^0 \times D^3 \to D^1 \times S^2$. On a connected manifold this surgery has the effect taking a connected sum with $S^2 \times S^1$.

It is possible to use similar tricks to get the remaining cases.

If A, B $\subset \Phi_6$, let $\langle A,B \rangle$ be the smallest index containing A and B (e.g. $\langle 36,126 \rangle = \{36, 26, 16, 6, 126\}$). Let $(j, k, \dots) = (G_0 \times S^1)^j \# (G_1 \times S^1)^k \# \dots$

The configuration spaces of Σ_6 are

$\varnothing$	$\varnothing$
$\{6\}$	S^3
$\langle 16 \rangle$	(1)

<26>	(2)
<36>	(3)
<46>	(4)
<56>	(5)
<126>	(0,1)
<126,36>	(1,1)
<126,46>	(2,1)
<126,56>	(3,1)
<136>	(0,0,1)
<136,46>	(1,0,1)
<136,56>	(2,0,1)
<146>	(0,0,0,1)
<146,56>	(1,0,0,1)
<156>	(0,0,0,0,1)
<1236>	$(0,1) \bigcup (0,1)$
<236>	(0,2)
<236,46>	(1,2)
<236,56>	(2,2)

Note that except for symmetries resulting from reordering the bars, all the cells in the above examples have distinct configuration spaces.

The methods used above will not work for $n \geqslant 7$; in these cases not every cell can be reached from a known cell via index-0 surgeries. We would still like to show that all the cells have distinct configuration spaces (except for symmetries). The homology (as a group) is not enough to distinguish the cells, so we compute the ring structure of the cohomology.

Since we have representative submanifolds for the generators of $H_*(C(\Sigma_n,\ell))$, it is natural to try to take intersections of these guys. In some cases, this works well. For example, if $A, B \subset \Phi_n$ and $A \bigcap B = \{m\}$, it is easy to find representatives $M \in r(A)$ and $N \in r(B)$ such that $M \bigcap N \in r(A \bigcup B)$ and M and N are transverse. If $A \bigcap B \neq \{m\}$ ($\Rightarrow$ it is bigger), we can find representatives M and N such that $M \bigcap N = \emptyset$. We conclude that

$$r(A)\ r(B) = \begin{cases} r(A \bigcup B), & A \bigcap B = \{m\} \\ \\ 0, & \text{otherwise} \end{cases}$$

The $f(A,s_A)$'s are harder to deal with, since they depend on the section s_A. So we switch to cohomology.

$H^*(C(\Sigma_n,l))$ is freely generated by

$$\{D(r(A)) \in H^{|A|}(C(\Sigma_n,l)) \ : \ A \in \Phi_n,\ \Sigma A < 1/2\}$$

and

$$\{D(f(A,s_A)) \in H^{(n-3)-|A|}(C(\Sigma_n,l)) \ : \ A \in \Phi_n,\ \Sigma A < 1/2\}$$

where $D : H_*(C(\Sigma_n,l)) \to H^*(C(\Sigma_n,l))$ is the Poincare duality isomorphism.

For $A \in \Phi_n$, let $T^A = (S^1)^{A-\{m\}}$ (= functions A-$\{m\} \to S^1$). Define

$$\pi_A : C(\Sigma_n,l) \to T^A$$
$$p \to (\alpha(m,i)(p)), \quad i \in A-\{m\}$$

For each $A \in \Phi_n$ such that $\Sigma A < 1/2$, pick a section s_A of π_A. If $B \subset A$, let π_B^A be the restriction.

Note that $r(A) = [\pi_A^{-1}(pt)]$ and $f(A,s_A) = s_{A*}(\omega_A)$, where ω_A is a generator of $H_{|A|-1}(T^A)$. Thus $f^*(A) := D(f(A,s_A)) = D(s_{A*}(\omega_A)) = D(s_{A*}(D^{-1}(1_A))) = s_A^!(1_A)$, where 1_A is a generator of $H^0(T^A)$. Similarly, $r^*(A) := D(\pi_A^{-1}(pt)) = \pi_A^*(\omega_A^*)$, where ω_A^* is a generator of $H^{|A|-1}(T^A)$. (Note that $\Sigma A > 1/2 \Rightarrow \pi_A$ is not surjective $\Rightarrow \pi_A^*(\omega_A^*) = 0$.)

Since we won't be dealing with homology anymore, write $r(A)$, $f(A)$ and ω_A for $r^*(A)$, $f^*(A)$ and ω_A^*.

Proposition 3.6. It is possible to choose sections $\{s_A\}$ such that for all $A, B \in \Phi_n$, $\Sigma A, \Sigma B <$ $1/2$

$$r(A)r(B) = \begin{cases} \pm r(A \bigcup B), & A \bigcup B = \{m\} \\ 0, & \text{otherwise} \end{cases}$$

$$r(A)f(B) = \begin{cases} \pm f(\cancel{A-B}\ B-A), & \cancel{\text{otherwise}}\ A \subset B \\ 0, & \text{otherwise} \end{cases}$$

$$f(A)f(B) = \begin{cases} \pm f(A \bigcap B), & |A \bigcup B| = n-2 \end{cases}$$

$\left\{ 0, \quad \text{otherwise} \right.$

Proof. Label the edges 1, 2, . . . , n, and assume $l(1) \leqslant l(2) \leqslant \dots < l(n)$.

Lemma (?) 3.7. There are sections $\{s_A\}$ ($A \in \Phi_n$, $\Sigma A < 1/2$) such that for any A and any i, j $\in \bar{A}$, $1 \leqslant i < j < n$, there is a map s'_A homotopic to s_A such that

$$0 < \alpha(i,j)(s'_A(x)) < \pi$$

for all $x \in T^A$. In addition, the map $\pi_N s_A$ is homotopic to the standard section $s_A^N : T^A \to T^N$, where $N = \{1, \dots, \hat{i}, \dots, \hat{j}, \dots, n\}$.

I have not been able to prove this. A sketch of one approach is given at the end.

Let $M = C(\Sigma_n, l)$. Let $A^- = A - \{m\}$.

We need some information about $H^*(T^A)$. For each $a \in A^-$, let $\theta_a = (\pi^A_{\{a,n\}})^*(\alpha) \in H^1(T^A)$, where α is a generator of $H^1(T^{\{a,n\}}) = H^1(S^1)$. Then $H^*(T^A)$ is the exterior algebra generated by $\{\theta_a : a \in A^-\}$. We can take $\omega_A = \theta_{a_1}\theta_{a_2}\cdots\theta_{a_k}$, where $A^- = \{a_1, \dots, a_k\}$. If $a \in B \subset A$, $\theta_a \; (\in H^1(T^A)) = (\pi^A_B)^*(\theta_a \; (\in H^1(T^B)))$.

Our strategy will be to pull the r's and f's back to some torus T^A and take the products there.

Consider first $r(A)r(B)$.

$r(A)r(B) = \pi_A^*(\omega_A)\pi_B^*(\omega_B)$

$$\begin{tikzcd} & M \arrow[dl, "\pi_A"'] \arrow[d, "\pi_{A\cup B}"] \arrow[dr, "\pi_B"] & \\ T^A & T^{A\cup B} \arrow[l, "\pi^{A\cup B}_A"] \arrow[r, "\pi^{A\cup B}_B"'] & T^B \end{tikzcd}$$

$$= \pi^{\cdot}_{A \cup B}(\pi_A^{A \cup B}(\omega_A)\pi_B^{A \cup B}(\omega_B))$$

$$= \pi^{\cdot}_{A \cup B}((\prod_{a \in A^-} \theta_a)(\prod_{b \in B^-} \theta_b)).$$

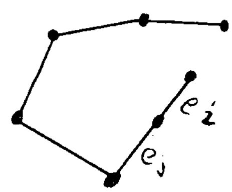
Since

$$(\prod_{a \in A^-} \theta_a)(\prod_{b \in B^-} \theta_b) = \begin{cases} \pm \omega_{A \cup B}, & A^- \cap B^- = \varnothing \\ 0, & \text{otherwise} \end{cases}$$

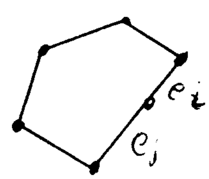
$$r(A)r(B) = \begin{cases} \pm r(A \cup B), & A \cap B = \{m\} \\ 0, & \text{otherwise} \end{cases}$$

Now for $r(A)f(B)$. Let $N \subset A \cup B$, $|N| = n-2$. (This is possible since $\Sigma A < 1/2$, $\Sigma B < 1/2$ and n is the longest edge.) Let $\{i,j\} = \bar{N}$, $i < j$. For each $x \in T^N$, $\pi_N^{-1}(x) \subset M$ consists of (i) 0 points, (ii) 1 point y, $\alpha(i,j)(y) = 0$ or π, or (iii) 2 points y_1 and y_2, $0 < \alpha(i,j)(y_1) < \pi < \alpha(i,j)(y_2) < 2\pi = 0$.

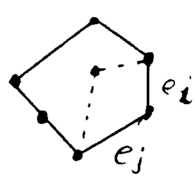
(i)



(ii)



(iii)



Let $U \subset T^N$ be the open set for which $\pi_N^{-1}(x) = 2$ points. Let $s_N : U \to M$ be the section of π_N such that $0 < \alpha(i,j)(s_N(x)) < \pi$ for all $x \in U$. Note that $s_N : U \to s_N(U)$ is a homeomorphism.

Let $C \subset N$, $n \in C$. By lemma 3.7, we may assume $0 < \alpha(i,j)(s_C(x)) < \pi$ for all $x \in T^C$. Therefore there is a unique map $\dot{s}_C^N : T^C \to U$ such that $s_N \dot{s}_C^N = s_C$.

Choose compact $K \subset U$ such that $\dot{s}_B^N(T^B), \dot{s}_{B-A}^N(T^{B-A}) \subset K$. Let $i : K \to T^N$ be the inclusion.

Write $\dot{\pi}_A^N$ for $\pi_A^N|_K$ and s_C^N for $i\,\dot{s}_C^N$, $C = B$ or $B-A$.

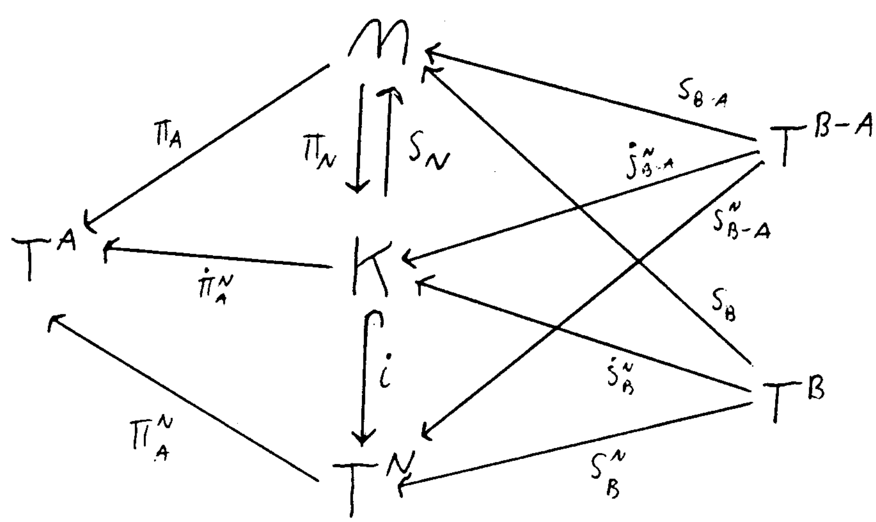
Here we go . . .

$$
\begin{aligned}
r(A)f(B) &= \pi_A^*(\omega_A)\, s_B^!(1_B) \\
&= \pi_N^*(\dot{\pi}_A^{N*}(\omega_A))\, s_N^!(\dot{s}_B^{N!}(1_B)) && \text{(by 0.3.1)} \\
&= s_N^!((s_N^*\pi_N^*)\dot{\pi}_A^{N*}(\omega_A)\, s_B^{N!}(1_B)) && \text{(by 0.3.2)} \\
&= s_N^! i^*(\pi_A^{N*}(\omega_A)\, s_B^{N!}(1_B)) && \text{(by 0.3.3)} \\
&= s_N^! i^*(\pm(\prod_{a\in A^-}\theta_a)(\prod_{b\in N-B}\theta_b)) && \text{(by lemma 3.7)}
\end{aligned}
$$

If $A \bigcap (N-B) \neq \emptyset \Leftrightarrow A \subset B$,

$$(\prod_{a\in A^-}\theta_a)(\prod_{b\in N-B}\theta_b)) = 0 \Rightarrow r(A)f(B) = 0.$$

Otherwise,



$$f(A)f(B) = s_N^! i^*(\pm \prod_{c\in N-(B-A)} \theta_c)$$

$$= \pm s_N^! i^*(s_{B-A}^{N!}(1_{B-A}))$$

$$= \pm s_N^! \dot{s}_{B-A}^{N!}(1_{B-A}) = \pm s_{B-A}^!(1_{B-A}) = \pm f(B-A)$$

The computation for $f(A)f(B)$ is similar.

$$f(A)f(B) = s_A^!(1_A)\, s_B^!(1_B)$$

$$= s_N^! i^*(s_A^{N!}(1_A)\, s_B^{N!}(1_B))$$

$$= s_N^! i^*(\pm (\prod_{a\in N-A} \theta_a)(\prod_{b\in N-B} \theta_b))$$

If $N\text{-}A \bigcap N\text{-}B \neq \emptyset \Leftrightarrow |A \bigcup B| < n\text{-}2$,

$$(\prod_{a\in N-A} \theta_a)(\prod_{b\in N-B} \theta_b)) = 0 \;\Rightarrow\; f(A)f(B) = 0$$

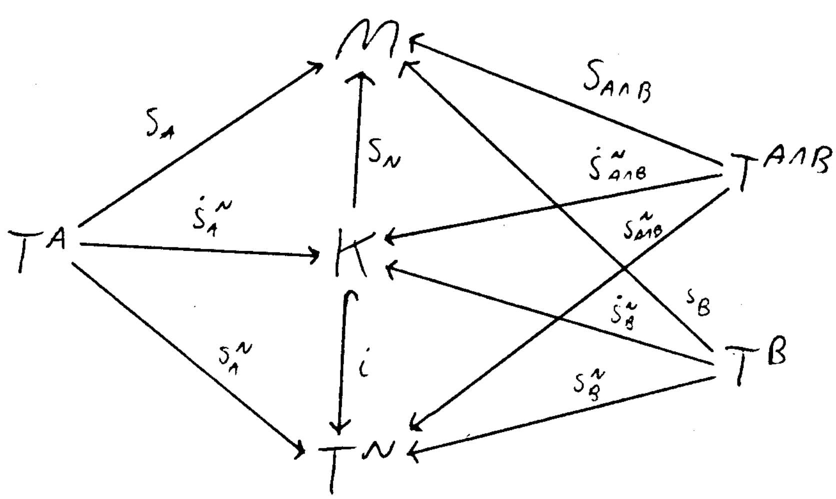
Otherwise,

$$f(A)f(B) = \pm s_N^! i^*(\prod_{c\in N-(A\bigcup B)} \theta_c)$$

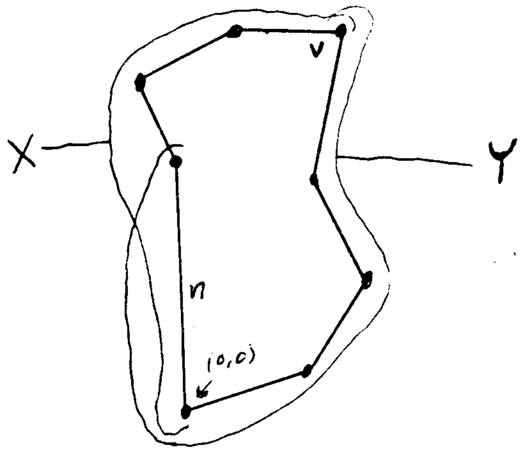
$$= \pm s_N^! i^*(s_{A\bigcap B}^{N!}(1_{A\bigcap B}))$$

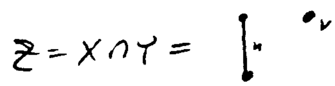
$$= \pm s_{A\bigcap B}^!(1_{A\bigcap B}) = \pm f(A \bigcap B)$$

All that remains is to prove lemma 3.7. Unfortunately, I do not have a satisfactory proof of this. Here's a sketch of an approach which seems promising.



Let $A = \{a_1, \ldots, a_j, n\}$, $\overline{A} = \{b_1, \ldots, b_k\}$, $a_1 > \ldots > a_j$, $b_1 > \ldots > b_k$. Let X, Y and Z be

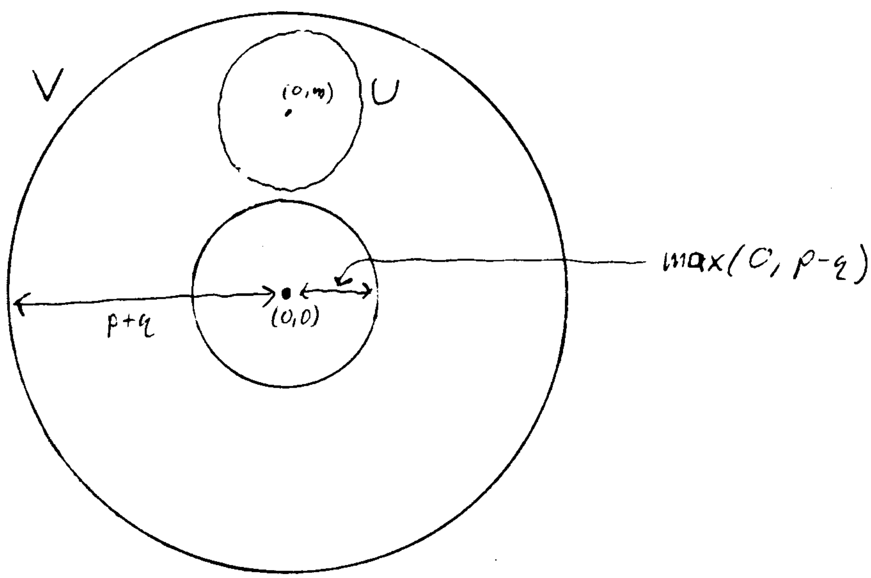




We must find a section of π^1_X. By 2.3, it suffices to find $f : C(X) \to C(Y)$ such that $\pi^X_Z \, \mathrm{id} = \pi^Y_Z f$.

Let $m = l(n)$, $p = l(b_1)$, $r = \sum l(a_i)$, $q = \sum_{i=2}^{k} l(b_i)$.

Parameterize C(Z) by the position of v, with the origin as shown above. $\pi^X_Z(C(Z))$ is constained in U := disk of radius r about (0,m). $\pi^Y_Z(C(Y))$ is the annulus/disk V :=



which contains U. It suffices to find a section $s : U \to C(Y)$, $\pi^Y_Z s = \mathrm{id}$, for then we can define $f = s\pi^X_Z$.

If we define s_A in this way, $\pi_N s_A$ will be homotopic to the standard section $T^A \to T^N$, because U is contractable.

To prove the rest of the lemma, it suffices to find s such that for any i, j $\in \overline{A}$, i < j, there is an s' vertically homotopic to s such that $0 < \alpha(i,j)(s'(x)) < \pi$ for all $x \in U$. It is not difficult to find a section which does this for fixed i and j. But I have not been able to prove that there is a homotopy class of sections which do the job for all i and j. $\boxtimes$

Conjecture. Cells not related by a permutation of $\{1, 2, \ldots, n\}$ have distinct cohomology rings, and hence distinct configuration spaces.

Bibliography

DOLD, A. : Lectures on Algebraic Topology. Berlin: Springer-Verlag 1972.

MILNOR, J. : Lectures on the h-Cobordism Theorem. Princeton: Princeton University Press 1965.

SPANIER, E.H. : Algebraic Topology. Berlin: Springer-Verlag 1966.

THURSTON, W.P., WEEKS, J.R. : The Mathematics of Three-Dimensional Manifolds. Scientific American 251, 94-107 (1985).